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PORTFOLIO OPTIMIZATION OF DYNAMIC COPULA MODELS FOR DEPENDENT FINANCIAL DATA USING CHANGE POINT APPROACH

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Abstract. In this paper, the portfolio optimization based on $CVaR$ is performed using the dynamic copula model for financial data. Determining the best model of dependency between financial data has an important role in taking appropriate investment decisions. Due to the financial data is always affected by the fluctuations of the economic factors, the dynamic model was handled. On the other hand change point detection is also important for investment decisions. So this study presents an application of dynamic copula model with change point approach. We take the currency data (USD and EUR) from Turkish Central Bank to construct a portfolio. This study consists of two stages. In the first stage, the marginal distributions and copula models of currency data are defined for full sample, and the portfolio optimization based on $CVaR$ is performed. In the second stage, the change periods of copula models are determined using binary segmentation method, and the portfolio optimization based on $CVaR$ is performed for each period.

1. INTRODUCTION

Dynamic copula modeling is commonly used in finance and risk management. This approach is important in practice when financial data don't have normal distributions and have high volatility by time-varying methods. In today's complex financial markets, there are dependencies between assets constituting the finance portfolio. Therefore, the relationship between the returns of the assets may not have a linear correlation. For this purpose, dependencies between financial assets can be modeled with the dynamic copula approach by taking into account time-variations.

Furthermore, in the long term, the returns on assets can be affected by some changes of politics or economic factors. Therefore, the determination of the change points is important to lead to the different portfolio selection for the various periods and to achieve more accurate risk calculations. So, the change point approach is used for determinating the change points.

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We applied portfolio optimization based on risk measures such as the Value at Risk (VaR) and the Conditional Value at Risk ($CVaR$) for assets modeled with dynamic copula at each specified period by using the change point approach. In order to conduct a portfolio optimization, firstly we determined the period. Then, we chose the appropriate copula model for every period by using AIC [1] and BIC [2]. The inference function for margins (IFM) method was used to estimate copula parameters. Finally, portfolio optimizations based on the $CVaR$ were performed for each period using data obtained from the Monte Carlo simulation method.

The copula method in financial risk management has been introduced by Embrechts et al. [3]. Further details on copula can be seen in Joe [4], Nelsen [5] and Cherubini [6]. Some studies on dynamic copula are found in Wei and Zhang [7], Ozun and Cifter [8], Jondeau and Rockinger [9] and Huang et al. [10]. On the other hand, Rockafellar and Uryasev [11] studied the portfolio optimization based on $CVaR$. The applications of the dynamic copula model for portfolio analysis on risk measures were given in Wu et al. [12], Wang et al. [13] and He and Li [14]. The change points approach has been introduced by Gombay and Horváth [15], [16] and Csörgö and Horváth [17]. One of the latest studies in finance based on copula with change point detection was given in Zhu et al. [18]. Finally, Dias and Embrechts [19], [20] and Guegan and Zhang [21] have studied dynamic copula models in finance and insurance by taking into account the change point approach.

In this study, we give an application of portfolio optimization based on the change point detection approach that provides better portfolio selection and investment decisions when dependent financial data are modeled with the dynamic copula. The rest of the paper is organized as follows. In Section 2, the expressions of copula, dynamic copula modeling and parameter estimations are presented. The usage of the change point approach is presented in Section 3 and the general steps are given. Portfolio optimization based on $CVaR$ is defined in Section 4. Section 5 presents a useful application of this approach. Finally, some conclusions are given in Section 6.

2. DYNAMIC COPULA MODEL AND PARAMETER ESTIMATION

To define bivariate copula, suppose $F(x_1, x_2)$ is a joint distribution with corresponding marginal distributions $F_1(x_1)$ and $F_2(x_2)$. Then $F(x_1, x_2)$ can be expressed as

$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2)) \tag{2.1}$$

where C is a parametric copula function which we know to exist uniquely by Sklar's Theorem [22].

The static copula models, Gaussian, Student-t, Clayton and Symmetrized Joe Clayton-SJC and corresponding dynamic copula models, GDCC, tDCC, tvC and tvSJC are used in this study which are given in Appendix. Dynamic copula models are utilized for modeling of the financial data varying to the time and it is defined as

$$F\left(X_{1t},...,X_{nt}\mid\xi_t\right)=C_t\left(F_{1t}\left(X_{1t}\mid\xi_t\right),F_{2t}\left(X_{2t}\mid\xi_t\right)...,F_{nt}\left(X_{nt}\mid\xi_t\right)\right) \tag{2.2}$$

where $\xi_t=\sigma\left\{X_{1t-1},X_{2t-2},...,X_{nt-t},...\right\},\ t=1,2,...,T$ represents the historical data until t time [8].

Portfolio assets based on the dynamic copula method are generally modelled with the GARCH or GJR models [23]. GARCH-n and GARCH- t models are defined as

$$\begin{aligned} x_t &= \mu+a_t \\ a_t &= \sigma_t\varepsilon_t \\ \sigma_t^2 &= \alpha_0+\alpha_1 a_{t-1}^2+\beta\sigma_{t-1}^2 \\ & \varepsilon_t\sim N\left(0,1\right)\ \text{ or } \varepsilon_t\sim t_d \end{aligned} \tag{2.3}$$

Here, provided that μ indicates the conditional mean of return series, and σ_{t-1}^2 indicates the conditional variance, d indicates the degree of freedom.

GJR-n and GJR- t models are defined as:

$$\begin{aligned} x_t &= \mu+a_t \\ a_t &= \sigma_t\varepsilon_t \\ \sigma_t^2 &= \alpha_0+\alpha_1 a_{t-1}^2+\beta\sigma_{t-1}^2+\gamma s_{t-1}a_{t-1}^2 \\ & \varepsilon_t\sim N\left(0,1\right)\ \text{ or } \varepsilon_t\sim t_d \end{aligned} \tag{2.4}$$

where,

$$s_{t-1}=\begin{cases} 1, & a_{t-1}<0 \\ 0, & a_{t-1}\geq 0 \end{cases},\alpha_0>0,\alpha_1\geq 0,\beta\geq 0,\ \beta+\gamma\geq 0$$

and

$$\alpha_1+\beta+\frac{1}{2}\gamma<1.$$

The parameters of the GARCH and GJR models are estimated with the MLE method. The joint density function can be expressed as

$$f\left(a_1,...,a_t\right)=f\left(a_t|\Omega_{t-1}\right)f\left(a_{t-1}|\Omega_{t-2}\right)...f\left(a_1|\Omega_0\right)f\left(a_0\right)$$

where $\Omega_{t-1}=\left\{a_0,a_1,...,a_{t-1}\right\}$. Given data $a_1,...,a_t$, the log-likelihood function is given as:

$$LL=\sum_{k=0}^{n-1}f\left(a_{n-k}|\Omega_{n-k-1}\right)$$

The copula parameters are estimated by the Inference Function for Margins (IFM) method. The estimation procedure of the IFM method consists of two steps [6].

In the first step, the parameters of the marginals are estimated as

$$\widehat{\theta}_1=\arg\max\theta_1\sum_{t=1}^{T}\sum_{j=1}^{n}\ln f_j\left(x_{jt};\theta_1\right). \tag{2.5}$$

In the second step, the parameter of the copula model is estimated, given θ_1:

$$\widehat{\theta}_2 = \arg\max \theta_2 \sum_{t=1}^{T} \ln c \left(F_1(x_{1t}), F_2(x_{2t}), ..., F_n(x_{nt}); \theta_2, \widehat{\theta}_1 \right). \tag{2.6}$$

The IFM estimator is defined as [6]

$$\widehat{\theta}_{IFM} = \left(\widehat{\theta}_1, \widehat{\theta}_2 \right)' \tag{2.7}$$

AIC (Akaike Information Criterion) [1] and BIC (Bayesian Information Criterion) [2] are used to make the goodness of fit tests for marginal distributions and copula models.

$$AIC = -2 * LL + 2k \tag{2.8}$$

$$BIC = -2 * LL + \ln(n) * k \tag{2.9}$$

where LL is the log-likelihood at its maximum point of the model estimated and k is the number of copula parameters in the model. The model having the smallest AIC and BIC value is the best choice.

3. CHANGE POINT DETECTION

The change point approach is a procedure used to determine the change time. At first, an appropriate copula is determined by making a model selection on the whole sample. For the dynamic copula model, it is important to test the stability of the dependence structure. For this purpose, it needs to apply goodness-of-fit (GOF) test proposed by Genest et al. [24] to test whether the selected copula is stable. If the copula does not change, it is dealt with the changes of copula's parameters for same copula family. In this case, it is applied for the change-point analysis as given in Csörgö and Horvath [17], Dias and Embrechts [20]. If there is a change in the copulas, then the binary segmentation procedure proposed by Vostrikova [25] can be applied to detect the change time. In this procedure, the whole sample is divided into two subsamples, then the best copula family is chosen for each subsample by AIC. The procedure will continue with binary segmentation until two subsamples have the same best fit copula. Finally, all the change points according to copula family and parameters will be detected. To detect changes of copula parameters, we use the following procedure [14], [19].

Let $u_1, u_2, ..., u_n$ be a sequence of independent random vectors in $[0,1]^d$ with univariate uniformly distributed margins and copulas

$$C(u; \theta_1, \eta_1), C(u; \theta_2, \eta_2), ..., C(u; \theta_n, \eta_n)$$

respectively, where θ_i and η_i represent the dynamic and static copula parameters. To test the null hypothesis

$$H_0 : \theta_1 = \theta_2 = \cdots = \theta_n \text{ and } \eta_1 = \eta_2 = \cdots = \eta_n$$

against alternative hypothesis

$$H_1 : \theta_1 = \cdots = \theta_{k^*} \neq \theta_{k^*+1} \cdots = \theta_n \text{ and } \eta_1 = \eta_2 = \cdots = \eta_n.$$

If the null hypothesis is rejected then k^* denotes the change point. If $k^* = k$, the null hypothesis would be rejected for small values of the generalized likelihood ratio

$$\Lambda_k = \frac{\sup_{(\theta,\eta)\in\Theta^{(1)}x\Theta^{(2)}} \prod_{1\leq i\leq k} c(u_i;\theta,\eta)}{\sup_{(\theta,\theta',\eta)\in\Theta^{(1)}x\Theta^{(2)}} \prod_{1\leq i\leq k} c(u_i;\theta,\eta) \prod_{k\leq i\leq n} c(u_i;\theta',\eta)} \tag{3.1}$$

where c is the density of copula C. H_0 will be rejected for large values of

$$Z_n = \max_{1\leq k\leq n} (-2\log(\Lambda_k))$$

According to Csörgö and Horvath [17], the following approximation holds

$$P(Z_n^{1/2} \geq x) \approx \frac{x^p \exp(-x^2/2)}{2^{p/2}\Gamma(p/2)} \left(\log\frac{(1-h)(1-l)}{hl} - \frac{p}{x^2}\log\frac{(1-h)(1-l)}{hl} + \frac{4}{x^2} + O(\frac{1}{x^4})\right)$$

where h and l can be taken as $h(n) = l(n) = (\log n)^{3/2}/n$ as $x \longrightarrow \infty$. If there is exactly one change point then the maximum likelihood estimator for the change point is given by $\widehat{k}_n = \min\{1 \leq k \leq n : Z_n = -\log(\Lambda_k)\}$ [14], [19].

4. PORTFOLIO OPTIMIZATION BASED ON CVAR

Assuming that $x = (x_1, x_2, ..., x_n)^T$ is a portfolio vector for n assets and $y = (y_1, y_2, ..., y_m)^T$ is the m type loss factor with $p(y)$ density distribution. Rockafellar and Uryasev [11] defined a simple convex optimization problem based on composition of VaR and $CVaR$ and this determination is given for $\beta \in (0, 1)$ confidence level by following function

$$F_\beta(x,\alpha) = \alpha + \frac{1}{1-\beta}\int_{y\in R^m} [f(x,y)-\alpha]^+ p(y)dy. \tag{4.1}$$

Here $\alpha_\beta(x)$ and $\Phi_\beta(x)$ indicate VaR and $CVaR$ respectively as follows:

$$\begin{aligned} \alpha_\beta(x) &= \min\{\alpha \in R : \Psi(x,\alpha) \geq \beta\} \\ \Phi_\beta(x) &= E[f(x,y)|f(x,y) \geq \alpha_\beta(x)] \end{aligned}$$

$\widetilde{F}_\beta(x,\alpha)$ can be used instead of $F_\beta(x,\alpha)$ for the derived returns y by Monte Carlo simulation as

$$\widetilde{F}_\beta(x,\alpha) = \alpha + \frac{1}{m(1-\beta)}\sum_{j=1}^{m}\left[x^T y_j - \alpha\right]^+ \tag{4.2}$$

where m denotes the number of simulations. Let $x = (x_1, x_2, ..., x_n)$ indicates the weights of assets in the portfolio and $y_j = (y_{j1}, y_{j2}, ..., y_{jn})$ indicates the derived returns. The $CVaR$ optimization problem is given by

$$\begin{aligned} \min \widetilde{F}_\beta(x,\alpha) &= \min\left(\alpha + \frac{1}{m\,(1-\beta)}\sum_{j=1}^{m} z_j\right) \\ z_j &= \left[-x^T y_j - \alpha\right]^+ \\ & \left\{ \begin{array}{c} x^T y_j + \alpha + z_j \geq 0 \\ z_j \geq 0 \\ \frac{1}{q} x^T \sum y_j \geq \rho \\ \sum_{i=1}^{m} x_j = 1, \quad x \geq 0 \end{array} \right. \end{aligned} \tag{4.3}$$

where ρ is the expected return. It can be solved as a linear programming problem.

5. NUMERICAL EXAMPLE

In this study, 1007 daily USD and EUR currency data between January 3, 2011 and December 31, 2014 were used as an application. USD and EUR currency data were taken from the website of the Turkish Central Bank [26]. The volatility clustering of daily market returns of USD and EUR are illustrated in Figure 1.

The returns, r_t are calculated by

$$r_t = 100 \times \ln\left(\frac{P_t}{P_{t-1}}\right)$$

where P_t indicates the index value at t time.

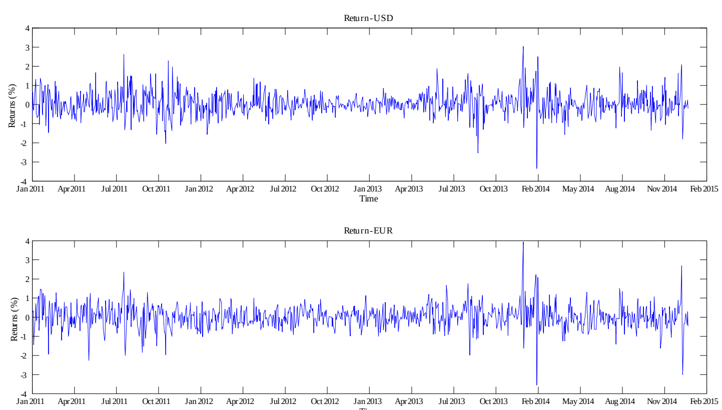

Figure 1. Daily returns of USD and EUR

Descriptive statistics and ARCH effects test results of currency data are given in Table 1. It can be seen that the marginals of USD and EUR are not distributed normally according to JB statistics ($p\ value < 0.05$). Due to the Engle test results (LM statistics) there are ARCH effects ($p\ value < 0.05$). Furthermore, the correlations for returns in Figure 2 were evaluated with autocorrelation functions (ACF) and partial autocorrelation functions (PACF). Here, it was observed that all autocorrelations were not zero after zero lag. Since financial returns are considered to be time series data, their marginal distributions were regarded as GARCH and GJR models.

Table 1. Descriptive statistics and tests

Statistics		USD		EUR	
Sample number		1007		1007	
Mean		0.0400		0.0314	
Standart deviation		0.6061		0.5945	
Skewness		0.2363		-0.0421	
Kurtosis		2.8009		4.6870	
Tests		Q-stat.	p-value	Q-stat.	p-value
Jarque-Bera		333.8726	0.0001	910.5830	0.0001
Engle-ARCH	LM(4)	85.6759	0.0000	100.2175	0.0000
	LM(6)	90.2060	0.0000	101.2061	0.0000
	LM(8)	96.7109	0.0000	101.1312	0.0000
	LM(10)	96.7643	0.0000	101.0680	0.0000

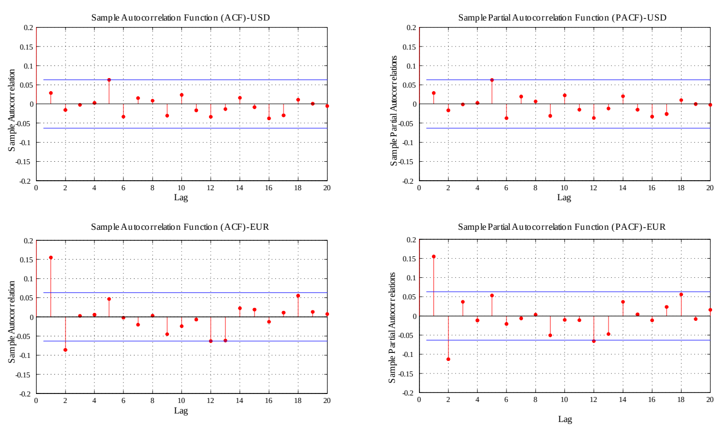

Figure 2. Autocorrelations USD and EUR indexes and partial autocorrelation measurements

It was determined by goodness of fit tests that the marginal distributions of the USD and the EUR is the best appropriated by GARCH (Normal, Student-t) or GJR (Normal, Student-t) models. Table 2 and Table 3 show that maximum likelihood results, estimated parameter values, AIC (Akaike Information Criterion) and BIC (Bayesian Information Criterion) values for GARCH and GJR models selection. It is demonstrated that both the USD and EUR can be modeled with GARCH-t since it has minimum AIC and BIC values.

In order to explain the dependency structure of USD and EUR, the best copula models were selected by using the estimated parameter values for the identified marginal distributions. The copula selection model was examined in two situations: static (Gaussian, Student-t, Clayton, SJC) and dynamic (GDCC, tDCC, tvC, tvSJC). Table 4 and Table 5 demonstrate the AIC and BIC values besides the estimated parameter results by the IFM method for these situations. Thus, the Gaussian copula and tDCC copula are selected for static and dynamic cases, respectively, since they have minimum AIC and BIC values. By using the parameter values for selected model, the Monte Carlo simulation was performed 10000 times, and portfolio optimization was achieved based on $CVaR$. VaR and $CVaR$ risk measurement values for different confidence levels, and results including the weights for each financial asset are given in Table 6. Accordingly, the $CVaR$ value would reach the minimum when an investor allocates of his/her assets 37% in USD and 63% in EUR within a 99% confidence level.

Table 2. Parameter estimates of GARCH-n, and GARCH-t models and statistic tests

GARCH-n					GARCH-t				
	USD		EUR			USD		EUR	
Parameter	Value	Std	Value	Std	Parameter	Value	Std	Value	Std
μ	0.0202	0.015	0.0153	0.019	μ	0.0104	0.014	0.0250	0.017
α_0	0.0058	0.006	0.0367	0.015	α_0	0.0047	0.004	0.0367	0.018
α_1	0.1102	0.049	0.2080	0.066	α_1	0.1082	0.045	0.1732	0.064
β	0.8803	0.057	0.6932	0.092	β	0.8867	0.047	0.7200	0.107
					d	6.7220	1.266	7.6785	1.700
LL	-833.360		-822.816		LL	-812.607		-805.652	
AIC	1674.7207		1653.6311		AIC	**1635.2130**		**1621.3042**	
BIC	1694.3797		1673.2901		BIC	1659.7867		1645.8779	

Table 3. Parameter estimates of GJR-n and GJR-t models and statistic tests

GJR-n					GJR-t				
	USD		EUR			USD		EUR	
Parameter	Value	Std	Value	Std	Parameter	Value	Std	Value	Std
μ	0.0202	0.030	0.0153	0.556	μ	0.0104	0.016	0.0250	0.017
α_0	0.0058	0.0104	0.0367	0.911	α_0	0.0046	0.008	0.0367	0.033
α_1	0.1103	0.097	0.2080	1.445	α_1	0.1082	0.080	0.1732	0.072
β	0.8803	0.192	0.6932	6.236	β	0.8869	0.138	0.7200	0.209
γ	0.0000	0.201	0.0000	7.021	γ	0.0000	0.153	0.0000	0.161
					d	6.7169	2.155	7.6786	1.893
LL	-833.361		-822.816		LL	-812.606		-805.653	
AIC	1676.7211		1655.6319		AIC	1637.2130		1623.3050	
BIC	1701.2948		1680.2055		BIC	1666.7013		1652.7934	

Table 4. Parameter estimates for static copula families and model selection statistics

		Gaussian		Student-t		Clayton		SJC	
Parameter	ρ	0.0823 (0.019)	d	2.2950 (0.300)	ω	0.3545 (0.019)	λ	0.4413 (0.039)	0.4596 (0.035)
	d	0.9147 (0.022)							
LL		256.308		203.809		141.587		196.345	
AIC		**-508.6156**		-405.6178		-281.1738		-388.6895	
BIC		-498.7862		-400.7030		-276.2591		-378.8601	

Table 5. Parameter estimates for dynamic copula families and model selection statistics

		GDCC		tDCC		tvC		tvSJC	
Parameter	μ	0.0823 (0.019)	d	3.9987 (0.739)	ω	0.1419 (0.051)	λ	-0.7380 (0.279)	0.1631 (0.056)
	β	0.9147 (0.022)	α	0.0524 (0.014)	α	-0.7605 (0.283)	α	0.9651 (0.012)	1.1030 (0.278)
			β	0.9473 (0.015)	β	0.9523 (0.017)	β	-5.6345 (1.356)	0.8175 (0.083)
LL		256.308		274.546		200.593		256.543	
AIC		-508.6156		**-543.0916**		-395.1857		-501.0861	
BIC		-498.7862		-528.3475		-380.4415		-471.5977	

Table 6. The results of portfolio optimization based on $CVaR$ for full sample

Full Sample	GARCH(1,1)-t, tDCC			
β	VaR	$CVaR$	x_1-USD	x_2-EUR
%90	0.4078	0.8321	0.4245	0.5755
%95	0.6770	1.1460	0.4209	0.5791
%99	1.4307	1.9740	0.3716	0.6284

Table 7. Change point dates and events

Periods	Dates	Events	Change Points
I	03.01.2011-14.12.2011		-
II	15.12.2011-14.05.2013	America withdrew from Iraq	15.12.2011
III	15.05.2013-14.07.2013	FED decision	15.05.2013
IV	15.07.2013-28.01.2014	TCMB announced that they may consider raising interest rates at MPC meeting	15.07.2013
V	29.01.2014-20.05.2014	MPC increased interest rates	29.01.2014
VI	21.05.2014-07.09.2014	Speech of FED Chairman	21.05.2014
VII	08.09.2014-31.12.2014	FED decreased interest rates.	08.09.2014

Table 7 presents change point dates (periods) and events which are obtained with the change point approach. The parameter estimates of GARCH (1,1)-t for periods are calculated and presented in Table 8. The estimates of static and dynamic copula parameters for each period are given in Table 9 and Table 10, respectively. Finally, portfolio optimization for these periods are carried out and the results are given Table 11.

Table 8. Estimate of GARCH (1,1)-t parameters for sample periods by change point approach

		Parameters				
Periods	Currency	μ	α_0	α_1	β	d
I	USD	0.0641	0.0302	0.0743	0.8680	28.6263
	EUR	0.0964	0.0524	0.0940	0.7905	7.0189
II	USD	-0.0056	0.0009	0.0467	0.9448	13.9576
	EUR	-0.0124	0.0067	0.0256	0.9332	172.7160
III	USD	0.1632	0.0303	0.0000	0.9315	3.6763
	EUR	0.2024	0.0079	0.0564	0.9436	39.3986
IV	USD	0.1030	0.0498	0.2106	0.7186	5.2203
	EUR	0.1265	0.0522	0.4250	0.5396	6.2406
V	USD	-0.1098	0.0587	0.1688	0.6834	5.6044
	EUR	-0.1083	0.0587	0.0885	0.7516	4.9476
VI	USD	-0.0060	0.0189	0.2406	0.7184	4.4100
	EUR	-0.0458	0.0229	0.1389	0.7656	5.5815
VII	USD	0.0391	0.0417	0.0000	0.9128	3.4960
	EUR	-0.0158	0.0417	0.0000	0.9031	3.3790

Table 9. Estimates of static copula parameters by change points approach

	Periods	Static	Change Time
I	03.01.2011 14.12.2011	$Gaussian$ $\rho: 0.0375(0.028)$ $d: 0.9320(0.033)$	-
II	15.12.2011 14.05.2013	$Gaussian$ $\rho: 0.0232(0.007)$ $d: 0.9767(0.007)$	15.12.2011
III	15.05.2013 14.07.2013	$Student-t$ $d: 3.0549(1.839)$	15.05.2013
IV	15.07.2013 28.01.2014	$Student-t$ $d: 2.1228(0.779)$	15.07.2013
V	29.01.2014 20.05.2014	$Student-t$ $d: 2.0100(4.265)$	29.01.2014
VI	21.05.2014 07.09.2014	$Student-t$ $d: 2.0102(10.723)$	21.05.2014
VII	08.09.2014 31.12.2014	$Student-t$ $d: 2.2983(1.050)$	08.09.2014

Table 10. Estimates of dynamic copula parameters by change points approach

	Periods	Dynamic	Change Time
I	03.01.2011 14.12.2011	*GDCC* $\mu : 0.0375(0.028)$ $\beta : 0.9320(0.033)$	-
II	15.12.2011 14.05.2013	*tDCC* $d : 8.4731(5.300)$ $\alpha : 0.0209(0.008)$ $\beta : 0.9763(0.006)$	15.12.2011
III	15.05.2013 14.07.2013	*GDCC* $\mu : 0.5000(0.286)$ $\beta : 0.1137(0.121)$	15.05.2013
IV	15.07.2013 28.01.2014	*tDCC* $d : 3.4672(2.412)$ $\alpha : 0.2425(0.165)$ $\beta : 0.2442(0.288)$	15.07.2013
V	29.01.2014 20.05.2014	*tDCC* $d : 2.7274(0.455)$ $\alpha : 0.0222(0.028)$ $\beta : 0.8308(0.071)$	29.01.2014
VI	21.05.2014 07.09.2014	*tDCC* $d : 2.5263(0.641)$ $\alpha : 0.0290(0.040)$ $\beta : 0.8308(0.071)$	21.05.2014
VII	08.09.2014 31.12.2014	*GDCC* $\mu : 0.0000(0.257)$ $\beta : 0.4958(4.044)$	08.09.2014

Table 11. The results of portfolio optimization based on $CVaR$ for change points

$\beta = \%95$	GARCH(1,1)-t, tDCC			
Periods	VaR	$CVaR$	$x_1 - USD$	$x_2 - EUR$
I	0.9434	1.2783	0.4080	0.5920
II	0.6109	0.8201	0.2104	0.7896
III	0.4170	0.5840	0.8215	0.1785
IV	1.1101	1.6882	0.5569	0.4431
V	0.0500	6.3605	0.1051	0.9949
VI	1.0813	1.4849	0.2442	0.7558
VII	0.5386	0.6853	0.4612	0.5388

6. CONCLUSION AND DISCUSSION

In this paper, we constructed a dynamic copula model for currency data (USD and EUR) with the change point approach. This study was conducted in two cases: Full sample and subsamples by change point aprroach. In the first case, marginal distributions were determined as GARCH-t and the best copula describing the dependence structure of currency data was found as Gaussian in the static model and tDCC in the dynamic model. In the second case, the full sample was divided into subsamples (periods) with the change point approach. This approach allows us to determine the change points for the types of copulas using the binary segmentation method. Thus, seven periods are obtained to study on. Marginal distributions and appropriate copula models are determined for each period. Then, optimization based on $CVaR$ by using Monte Carlo simulation was performed for full sample and seven periods. As result, it was seen how the copula models and their parameters changed from period to period. Therefore, the portfolio optimization results were obtained for each period and it was found the best way to allocate portfolio assets. As it can be inferred from the results of the numerical example, copula models may subject to change according to some political or financial events during the time schedule which may affect the future structure of investment strategies.

In future work, it is possible to study other copula families for another financial investment instruments by taking into account the change point approach.

7. APPENDIX

The used static copulas in this paper are given as follows:

Gaussian Copula
Defining $x = \phi^{-1}(u)$ and $y = \phi^{-1}(v)$, the Gaussian copula is given by

$$C_{Gau}(u, v; \rho) = \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}) dxdy$$

with $\rho \in (-1, 1)$, where ϕ^{-1} is the inverse of the normal c.d.f. and ρ is the coefficient of linear correlation. The Gaussian copula has no tail dependence. The Clayton copula is also Archimedean, but it is not rotationally symmetric and it only allows for positive dependence [5,6,16].

Student t Copula
Defining $x = t_d^{-1}(u)$ and $y = t_d^{-1}(v)$, the Student-t copula is given by

$$C_{Student}(u, v; \rho, d) = \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp(1 + \frac{x^2 - 2\rho xy + y^2}{d(1-\rho^2)}) dxdy$$

where t_d^{-1} is the inverse of a Student-t c.d.f., the parameters $\rho \in (-1,1)$ and $d \in (0,\infty)$ are the coefficient of linear correlation and the degrees of freedom, respectively [5,6,16].

Clayton copula

The Clayton copula is

$$C_{Clayton}(u,v;w) = (u^{-w} + v^{-w} - 1)^{-\frac{1}{w}}$$

with $w \in (0,\infty)$. The Clayton copula is also Archimedean, but it is not rotationally symmetric and it only allows for positive dependence [5,6].

The symmetrized Joe-Clayton (SJC) Copula

The Joe-Clayton copula is constructed by taking a particular Laplace transformation of Clayton's copula by Joe [4] and it is given by

$$C_{JC}\left(u,v|\tau^U,\tau^L\right) = 1 - \left(1 - \left\{[1-(1-u)^\kappa]^{-\gamma} + [1-(1-v)^\kappa]^{-\gamma} - 1\right\}^{-1/\gamma}\right)^{1/\kappa}$$

with $\kappa = \frac{1}{\log_2(2-\tau^U)}$ and $\gamma = \frac{1}{\log_2(\tau^L)}$. The symmetrized Joe-Clayton (SJC) copula is improved by Patton [27], in order to overcome the asymmetry property of the Joe-Clayton copula, when $\tau^U = \tau^L$, and this copula is defined by

$$C_{SJC}(u,v;w) = \frac{1}{2}\left(C_{JC}\left(u,v|\tau^U,\tau^L\right) + C_{JC}\left(1-u,1-v|\tau^L,\tau^U\right) + u + v - 1\right)$$

which is symmetric when $\tau^U = \tau^L$. The parameters $\tau^U \in (0,1)$ and $\tau^L \in (0,1)$ are the coefficients of upper and low tail dependence, respectively.

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