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A Dirichlet Problem For Generalized Analytic Functions

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Summary. For the existence of the solution for the Dirichlet Problem

$$\frac{\partial w}{\partial \overline{z}} = -(Aw + B\overline{w}), \quad z \in D$$
$$Rew|_{\partial D} = g, \quad g \in C^{\alpha}(\partial D)$$
$$Imw(z_0) = c_0, \quad z_0 \in \overline{D}$$

in a domain having a smooth boundary $D \subset C$, necessary conditions are studied. Here we assumed that $z \in D$, $g \in C^{\alpha}(\partial D)$, $z_0 \in \overline{D}$ and $A, B \in C^{\alpha}(\overline{D})$.

Key words: Generalized analytic functions, Dirichlet Problem, Contractive mapping, Boundary value problem.

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1. Introduction

Various boundary value problems for Generalized Analytic Functions have been studied by many authors [3], [5]. For instance, in [3] the function space has been changed and in [5] the boundary values of the problem have been given in various forms. In [4] , Tutschke presented the existence of a Dirichlet problem using fixed point theorem. However, these problems have been considered in various meanings. The purpose of this paper is to study the boundary value problems by putting special conditions on the coefficients of the equations. We consider the system of non-homogenous real-valued partial differential equation

$$\begin{aligned} u_x - v_y + au + bv = 0 \\ u_y + v_x + cu + dv = 0 \end{aligned} \tag{1}$$

It can be easily shown that (1) is equal to the complex-valued partial differential equation

$$\frac{\partial \omega}{\partial \overline{z}} = A\omega \; + \; B\overline{\omega} \tag{2}$$

with

$$A = \frac{1}{4}(a - d + ic + ib), B = \frac{1}{4}(a + d + ic - ib), w = u + iv.$$

2. A Dirichlet Problem For Generalized Analytic Functions

In this section, we will study the solution of the complex partial differential equation

$$\frac{\partial w}{\partial \overline{z}} \; + \; A(z)w \; + \; B(z)\overline{w} \; = \; 0 \qquad in \quad D \tag{3}$$

which belongs to the class of $C^\alpha(\overline{D})$, with the following boundary conditions:

$$Re\, w(z) \; = \; g, \qquad z \in \partial D \tag{4}$$

$$Im\, w(z_0) \; = \; c_0, \qquad z_0 \in \overline{D} \tag{5}$$

where D is a bounded and simply connected domain with smooth boundary, c_0 is a real constant, $A, B \in C^\alpha(\overline{D})$, and $g \in C^\alpha(\partial D)$ is Hölder continuous with Hölder constant H.

Now, by assuming $h \in C^\alpha(\overline{D})$, let us consider the operator T_D such that

$$\begin{aligned} T_D : C^\alpha(\overline{D}) \rightarrow C^\alpha(\overline{D}) \\ h \rightarrow T_D h(z) = -\frac{1}{\pi} \int_D \int \frac{h(\zeta)}{\zeta - z} d\xi d\mu \end{aligned}$$

where $\zeta = \xi + i\mu$.

Theorem 1: A function $w \in C^{1,\alpha}(\overline{D})$ is a solution to the Dirichlet problem (3)-(5) if and only if w solves the integral equation

$$w(z) = \varphi(z) + T_D[-(Aw + B\overline{w})] \tag{6}$$

where $\varphi \in C^\alpha(\overline{D})$ is holomorphic in a domain D satisfying the Dirichlet conditions

$$Re\varphi(z) = g - ReT_D[-(Aw + B\overline{w})](z), \qquad z \text{ on } \partial D \tag{7}$$

$$Im\varphi(z_0) = c_0 - ImT_D[-(Aw + B\overline{w})](z_0), \qquad z_0 \in \overline{D} \tag{8}$$

Proof: Assume $w \in C^{1,\alpha}(\overline{D})$ is a solution of (3) satisfying the boundary conditions (4) and (5). We define a function φ as follows:

$$\varphi(z) = w(z) - T_D[-(Aw + B\overline{w})].$$

Differentiating φ with respect to $\overline{z}$, we get

$$\frac{\partial \varphi}{\partial \overline{z}} = \frac{\partial w}{\partial \overline{z}} + Aw + B\overline{w} = 0$$

at least in Sobolev's sense. It follows from Weyl lemma that φ is a holomorphic function in D, hence the boundary conditions (7) and (8) implies:

$$Re\varphi(z) = Re[w - T_D(-Aw - B\overline{w})](z)$$

$$= g - ReT_D(-Aw - B\overline{w})(z)$$

and

$$Im\varphi(z_0) = Imw[z_0] - ImT_D(-Aw - B\overline{w})](z_0)$$
$$= c_0 - ImT_D(-Aw - B\overline{w})(z_0)$$

for the holomorphic function φ. It is given that $g \in C^\alpha(\partial D)$. Moreover, $T_D[-(Aw + B\overline{w})] \in C^\alpha(\overline{D})$ for $w \in C^\alpha(\overline{D})$. It follows that, $ReT_D[-(Aw + B\overline{w})] \in C^\alpha(\partial D)$ in particular. Then $\varphi \in C^\alpha(\overline{D})$ and therefore, we have shown that, w solves the integral equation

$$w(z) = \varphi(z) + T_D[-(Aw + B\overline{w})]$$

where φ is holomorphic and satisfies the conditions (7) and (8).

Conversely, suppose that w is a solution of the equation (6) where φ is a holomorphic function satisfying (7) and (8). Differentiating (6) with respect to $\overline{z}$ we obtain

$$\frac{\partial w}{\partial \overline{z}} = 0 - (Aw + B\overline{w})$$

and obviously, w satisfies (7) and (8). This shows that $w \in C^{1,\alpha}(\overline{D})$ is a solution of (3)-(4).

Let us consider a function $w \in C^{1,\alpha}(\overline{D})$ and define an operator

$$P : C^\alpha(\overline{D}) \to C^\alpha(\overline{D}) \tag{9}$$

$$w \to P(w) = \varphi_{(w)} + T_D(-Aw - B\overline{w})$$

where $\varphi_{(w)}$ is a holomorphic function in D and is uniquely determined by the boundary conditions

$$(10) \qquad Re\varphi_{(w)}(z) = g - ReT_D[-(Aw + B\overline{w})] \qquad z \ on \quad \partial D$$

$$(11) \qquad Im\varphi_{(w)}(z_0) = c_0 - ImT_D[-(Aw + B\overline{w})](z_0), \qquad z_0 \in (\overline{D}).$$

Hence, $P(w)$ satisfies the boundary condition (7),(8) and if w is a fixed point of the operator P, that is

$$w = \varphi_{(w)} + T_D[-(Aw + B\overline{w})]$$

then w is a solution of (3)-(4).

Theorem 2: If A and B belongs to $C^\alpha(\overline{D})$ then the operator

$$P : C^\alpha(\overline{D}) \to C^\alpha(\overline{D})$$

defined by (10) is contractive if

$$\|A\|_{C^\alpha(\overline{D})} + \|B\|_{C^\alpha(\overline{D})} < \frac{1}{(K+1)\|T_D\|_{C^\alpha(\overline{D})}}$$

where K is a constant depending on α only.
Proof: Let us choose w_1 and w_2 in $C^{1,\alpha}(\overline{D})$. Then we have

$$P(w_1) = \varphi_{(w_1)} + T_D[-(Aw_1 + B\overline{w_1})]$$

$$P(w_2) = \varphi_{(w_2)} + T_D[-(Aw_2 + B\overline{w_2})]$$

where the holomorphic functions $\varphi_{(w_1)}$ and $\varphi_{(w_2)}$ are uniquely determined by the boundary conditions

$$(12) \qquad Re\varphi_{(w_1)}(z) = g - ReT_D[-(Aw_1 + B\overline{w_1})](z), \qquad z \ on \quad \partial D$$

$$(13) \qquad Im\varphi_{(w_1)}(z_0) = c_0 - ImT_D[-(Aw_1 + B\overline{w_1})](z_0), \qquad z_0 \in \overline{D}$$

and

$$(14) \qquad Re\varphi_{(w_2)}(z) = g - ReT_D[-(Aw_2 + B\overline{w_2})](z), \qquad z \ on \quad \partial D$$

(15) $$Im\varphi_{(w_2)}(z_0) = c_0 - ImT_D[-(Aw_2 + B\overline{w_2})](z_0), \qquad z_0 \in \overline{D}$$

respectively. Therefore we obtain
(16)
$$\begin{aligned} P(w_1) - P(w_2) &= (\varphi_{(w_1)} - \varphi_{(w_2)}) + T_D[-(Aw_1 + B\overline{w_1}) + (Aw_2 + B\overline{w_2})] \\ &= (\varphi_{(w_1)} - \varphi_{(w_2)}) + T_D[-A(w_1 - w_2) - B(\overline{w_1} - \overline{w_2})] \end{aligned}$$

where $\varphi_{(w_1)} - \varphi_{(w_2)}$ have the boundary values

$$\begin{aligned} Re(\varphi_{(w_1)} - \varphi_{(w_2)})(z) &= -ReT_D[-(Aw_1 + B\overline{w_1}) + (Aw_2 + B\overline{w_2})](z) \\ &= -ReT_D[-A(w_1 - w_2) - B(\overline{w_1} - \overline{w_2})], \qquad z \ on \quad \partial D \\ Im(\varphi_{(w_1)} - \varphi_{(w_2)})(z_0) &= -ImT_D[-A(w_1 - w_2) - B(\overline{w_1} - \overline{w_2})](z_0), \quad z_0 \in \overline{D} \end{aligned}$$

In order to show that P is contractive, we will compare the distance between the elements $w_1, w_2 \in C^{1,\alpha}(\overline{D})$ and their corresponding images $P(w_1)$ and $P(w_2)$. Thus we will need the corresponding estimates for the norms of

$$(\varphi_{(w_1)} - \varphi_{(w_2)}) \quad and \quad T_D[-A(w_1 - w_2) - B(-\overline{w_1} - \overline{w_2})].$$

We have

$$\begin{aligned} \|T_D[-A(w_1-w_2)-B(\overline{w_1}-\overline{w_2})]\|_{C^\alpha(\overline{D})} &\leq \|T_D\|_{C^\alpha(\overline{D})}\|A(w_1-w_2)+B(\overline{w_1}-\overline{w_2})\|_{C^\alpha(\overline{D})} \\ &\leq \|T_D\|_{C^\alpha(\overline{D})}[\|A(w_1 - w_2)\|_{C^\alpha(\overline{D})} + \|B(w_1 - w_2)\|_{C^\alpha(\overline{D})}] \\ &= \|T_D\|_{C^\alpha(\overline{D})}[\|A\|_{C^\alpha(\overline{D})}\|w_1 - w_2\|_{C^\alpha(\overline{D})} + \|B\|_{C^\alpha(\overline{D})}\|w_1 - w_2\|_{C^\alpha(\overline{D})}] \\ &= \|T_D\|_{C^\alpha(\overline{D})}[\|A\|_{C^\alpha(\overline{D})} + \|B\|_{C^\alpha(\overline{D})}]\|w_1 - w_2\|_{C^\alpha(\overline{D})}. \end{aligned}$$

We know

$$\|\varphi_{(w_1)} - \varphi_{(w_2)}\|_{C^\alpha(\overline{D})} =$$

$$max \left\{ Sup_{\overline{D}} \left|\varphi_{w_1} - \varphi_{w_2}\right|, \ Sup_{z_1 \neq z_2} \frac{\left|(\varphi_{(w_1)} - \varphi_{(w_2)})(z_1) - (\varphi_{(w_1)} - \varphi_{(w_2)})(z_2)\right|}{|z_1 - z_2|^\alpha} \right\}$$

Now we consider Dirichlet Problem defined for $\varphi_{(w_1)} - \varphi_{(w_2)}$ and investigate the behaviour of the real part of the function in ∂D .

$$|-ReT_D[-A(w_1 - w_2) - B(\overline{w_1} - \overline{w_2})](z_1) + ReT_D[-A(w_1 - w_2) - B(\overline{w_1} - \overline{w_2})](z_2)|$$

$$\leq |-T_D[-A(w_1-w_2)-B\left((\overline{w_1}) - (\overline{w_2})\right)](z_1)+T_D[-A(w_1-w_2)-B((\overline{w_1})-(\overline{w_2}))](z_2)|$$

$\leq \|T_D[-A(w_1 - w_2) - B(\overline{w_1} - \overline{w_2})]\|_{C^\alpha(\overline{D})}|z_1 - z_2|^\alpha$

$\leq \|T_D\|_{C^\alpha(\overline{D})}\|[-A(w_1 - w_2) - B(\overline{w_1} - \overline{w_2})]\|_{C^\alpha(\overline{D})}|z_1 - z_2|^\alpha.$

Therefore, the real part is Hölder continuous with the Hölder constant being not larger than

$$\|T_D\|_{C^\alpha(\overline{D})}[\|A\|_{C^\alpha(\overline{D})} + \|B\|_{C^\alpha(\overline{D})}]\|w_1 - w_2\|_{C^\alpha(\overline{D})}.$$

Then

(17)
$$|(\varphi_{(w_1)} - \varphi_{(w_2)})(z_1) - (\varphi_{(w_1)} - \varphi_{(w_2)})(z_2)| \leq k\|T_D\|_{C^\alpha(\overline{D})}[\|A\|_{C^\alpha(\overline{D})} + \|B\|_{C^\alpha(\overline{D})}]\|w_1 - w_2\|_{C^\alpha(\overline{D})}|z_1 - z_2|^\alpha,$$

where k is a constant defined by

$$k = \frac{2^{\alpha+3}}{cos(\frac{\alpha\pi}{2})}\left[\frac{2}{\alpha\pi}(1 + 2^\alpha) + 1\right].$$

And finally

(18)
$$\begin{array}{rcl} |(\varphi_{(w_1)} - \varphi_{(w_2)}(z)| & \leq & 2^\alpha k\|T_D\|_{C^\alpha(\overline{D})}[\|A\|_{C^\alpha(\overline{D})} + \|B\|_{C^\alpha(\overline{D})}]\|w_1 - w_2\|_{C^\alpha(\overline{D})} \\ & & +Sup_{\,\partial D}| - ReT_D[-A(w_1 - w_2) - B(\overline{w_1} - \overline{w_2})]| \\ & & +| - ImT_D[-A(w_1 - w_2) - B(\overline{w_1} - \overline{w_2})](z_0)| \end{array}$$

Here

$$Sup_{\,\partial D}| - ReT_D[-A(w_1 - w_2) - B(\overline{w_1} - \overline{w_2})]|$$

$$\begin{array}{rl} \leq & Sup_{\,\partial D}|T_D[-A(w_1 - w_2) - B(\overline{w_1} - \overline{w_2})]| \\ \leq & Sup_{\,\overline{D}}|T_D[-A(w_1 - w_2) - B(\overline{w_1} - \overline{w_2})]| \\ \leq & \|T_D[-A(w_1 - w_2) - B(\overline{w_1} - \overline{w_2})]\|_{C^\alpha(\overline{D})} \\ \leq & \|T_D\|_{C^\alpha(\overline{D})}(\|A\|_{C^\alpha(\overline{D})} + \|B\|_{C^\alpha(\overline{D})})\|w_1 - w_2\|_{C^\alpha(\overline{D})} \end{array}$$

and

$$| - ImT_D[-A(w_1 - w_2) - B(\overline{w_1} - \overline{w_2})]|$$

$$\begin{array}{rl} \leq & |T_D[-A(w_1 - w_2) - B(\overline{w_1} - \overline{w_2})]| \\ \leq & Sup_{\,\overline{D}}|T_D[-A(w_1 - w_2) - B(\overline{w_1} - \overline{w_2})]| \\ \leq & \|T_D[-A(w_1 - w_2) - B(\overline{w_1} - \overline{w_2})]\|_{C^\alpha(\overline{D})} \\ \leq & \|T_D\|_{C^\alpha(\overline{D})}[\|A\|_{C^\alpha(\overline{D})} + \|B\|_{C^\alpha(\overline{D})}]\|w_1 - w_2\|_{C^\alpha(\overline{D})}. \end{array}$$

Thus we obtain

$$
(19) \qquad |(\varphi_{(w_1)} - \varphi_{(w_2)}(z)| \leq (2^\alpha k + 2)\|T_D\|_{C^\alpha(\overline{D})}[\|A\|_{C^\alpha(\overline{D})} + \|B\|_{C^\alpha(\overline{D})}]\|w_1 - w_2\|_{C^\alpha(\overline{D})}.
$$

Consequently, using (17) and (19) we have the following estimate

$$
\|\varphi_{w_1} - \varphi_{w_2}\|_{C^\alpha(\overline{D})} \leq (2^\alpha k+2)\|T_D\|_{C^\alpha(\overline{D})}[\|A\|_{C^\alpha(\overline{D})} + \|B\|_{C^\alpha(\overline{D})}]\|w_1 - w_2\|_{C^\alpha(\overline{D})}.
$$

If we call $K = 2^\alpha k + 2$, we get

$$
\begin{aligned}
&\|P(w_1) - P(w_2)\|_{C^\alpha(\overline{D})} \\
&\quad \leq \|\varphi_{(w_1)} - \varphi_{(w_2)}\|_{C^\alpha(\overline{D})} + \|T_D[-A(w_1 - w_2) - B(\overline{w_1} - \overline{w_2}]\|_{C^\alpha(\overline{D})} \\
&\quad \leq (2^\alpha k + 2)\|T_D\|_{C^\alpha(\overline{D})}[\|A\|_{C^\alpha(\overline{D})} + \|B\|_{C^\alpha(\overline{D})}]\|w_1 - w_2\|_{C^\alpha(\overline{D})} \\
&\quad = (K + 1)\|T_D\|_{C^\alpha(\overline{D})}[\|A\|_{C^\alpha(\overline{D})} + \|B\|_{C^\alpha(\overline{D})}]\|w_1 - w_2\|_{C^\alpha(\overline{D})}.
\end{aligned}
$$

Consequently, the operator (9) is contractive if

$$
(K + 1)\|T_D\|_{C^\alpha(\overline{D})}[\|A\|_{C^\alpha(\overline{D})} + \|B\|_{C^\alpha(\overline{D})}] < 1
$$

$$
[\|A\|_{C^\alpha(\overline{D})} + \|B\|_{C^\alpha(\overline{D})}] < \frac{1}{(K + 1)\|T_D\|_{C^\alpha(\overline{D})}}.
$$

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