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DETERMINATION OF THE DEFLECTION FUNCTION OF A COMPOSITE CANTILEVER BEAM USING THEORY OF ANISOTROPIC ELASTICITY

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Abstract- In this paper, deflection function of an orthotropic cantilever beam subjected to point and distributed load are obtained using anisotropic elasticity. The deflections at the free end of the beam are calculated numerically using obtained formulas for different fiber directions.

1. INTRODUCTION

Among the major advantages of composite structures over conventional metal structures are their comparatively high strength-to-weight and stiffness-to-weight ratios. As a result, fiber reinforced composite materials have been gaining wide application in spacecraft construction and structural systems[1]. Therefore, some researchers have studied on composite beams. Karakuzu et al [2] have investigated elasto-plastic stress analysis in metal-matrix composite beam loaded uniformly or by a single force at the free end by using an analytical solution. Özcan [3] has investigated elasto-plastic stress analysis in steel fiber reinforced thermoplastic orthotropic cantilever beam subjected to single force at the free end of the beam. Ever et al [4] have obtained shear correction factor and deflection of a composite beam having I cross section.

In this study, the deflection function of an orthotropic composite cantilever is obtained by means of anisotropic elasticity. The deflection of the free end of the beam (i.e. at point $x=0$ and $y=0$) is calculated numerically, using obtained deflection function, for different fiber directions

2. SOLUTION OF ANISOTROPIC ELASTICITY

Stress-strain relations in anisotropic elasticity theory are given as[5];

$$\varepsilon_x = a_{11}\sigma_x + a_{12}\sigma_y + a_{16}\tau_{xy} \tag{1.a}$$

$$\varepsilon_y = a_{12}\sigma_x + a_{22}\sigma_y + a_{26}\tau_{xy} \tag{1.b}$$

$$\gamma_{xy} = a_{16}\sigma_x + a_{26}\sigma_y + a_{66}\tau_{xy} \tag{1.c}$$

If $Cos\theta$ and $Sin\theta$ are taken as m and n respectively in above equations, cofficients of a_{ij}'s are

$$
\begin{aligned}
a_{11} &= S_{11}m^4 + (2S_{12} + S_{66})m^2n^2 + S_{22}n^4 \\
a_{12} &= S_{12}(m^4 + n^4) + (S_{11} + S_{22} - S_{66})m^2n^2 \\
a_{22} &= S_{11}n^4 + (2S_{12} + S_{66})m^2n^2 + S_{22}m^4 \\
a_{16} &= (2S_{11} - 2S_{12} - S_{66})nm^3 - (2S_{22} - 2S_{12} - S_{66})n^3m \\
a_{26} &= (2S_{11} - 2S_{12} - S_{66})mn^3 - (2S_{22} - 2S_{12} - S_{66})m^3n \\
a_{66} &= 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})n^2m^2 - S_{66}(n^4 + m^4)
\end{aligned} \tag{2}
$$

and

$$S_{11} = \frac{1}{E_1}, \qquad S_{12} = -\frac{\nu_{12}}{E_1}, \quad S_{22} = \frac{1}{E_2}, \qquad S_{66} = \frac{1}{G_{12}} \tag{3}$$

In addition, strain components in the theory of elasticity are given as [6];

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{4.a}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \tag{4.b}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{4.c}$$

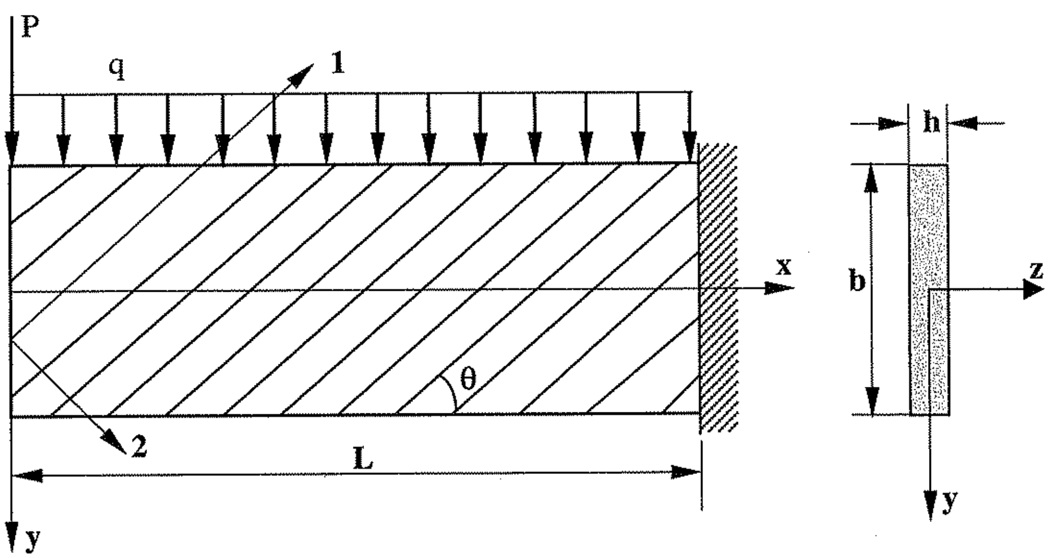

Figure 1. a) The Orthotropic cantilever beam subjected to single force (q=0)
b) The Orthotropic cantilever beam subjected to distributed load (P=0)

2.1. Deflection of The Orthotropic Cantilever Subjected to Single force

Stress components of cantilever beam subjected to single force are given as[5];

$$\sigma_x = -\frac{P}{I}xy + \frac{P}{I}\frac{a_{16}}{a_{11}}\left(\frac{b^2}{12} - y^2\right) \tag{5.a}$$

$$\sigma_y = 0 \tag{5.b}$$

$$\tau_{xy} = -\frac{P}{2I}\left(\frac{b^2}{4} - y^2\right) \tag{5.c}$$

where $I = \frac{hb^3}{12}$ (Fig. 1)

If equations of (5) are subsituted into Eq.(1a) and, this equation is equalized to Eq. 4a and is integrated as a function of x, the displacement function in the direction of x, is found;

$$u = -\frac{P}{I}\left[\frac{a_{11}}{2}x^2y + \frac{a_{16}(b^2+12y^2)}{24}x\right] + f(y) \tag{6}$$

In the same manner, if equations of (5) are subsituted into eq. (1b) and, this equation is equalized eq.(4b) and is integrated as a function of y, the displacement function in y direction is computed as;

$$v = \frac{P}{I}\left[\frac{(2a_{12}a_{16} - 3a_{11}a_{26})b^2}{24a_{11}}y - \frac{a_{12}}{2}xy^2 + \frac{(-2a_{12}a_{16} + a_{11}a_{26})}{6a_{11}}y^3\right] + g(x) \tag{7}$$

To find the displacement in y direction „ g(x) should be known. For that reason, if γ_{xy} in Eq.(1.c) is equalized to γ_{xy} in Eq.(4.c), then following equation is found;

$$\gamma_{xy1} - \gamma_{xy4} = \frac{P}{I}\left[\frac{a_{16}^2(b^2-12y^2)}{12a_{11}} - \frac{a_{66}b^2}{8} + \frac{a_{11}x^2 + a_{12}y^2 + a_{66}y^2}{2}\right] - f'(y) - g'(x) = 0 \tag{8}$$

Because of equality of eq.(8) to zero, the summution of the terms depending x and the summution of the terms depending y should be equal to separate constants. If c is constant then;

$$A(x) - g'(x) = c \tag{9}$$

Where $A(x) = \frac{P}{I}\frac{a_{11}}{2}x^2$ from the equation (9);

$$g(x) = \int [A(x) - c] dx \tag{10}$$

Integrating this eq.

$$g(x) = \frac{P}{I}\frac{a_{11}}{6}x^3 - cx + e \tag{11}$$

is found.

Substituting *g(x)* value into eq.(7) the deflection of the beam is found in terms oɪ c and e.

$$v = \frac{P}{I}\left[\frac{(2a_{12}a_{16} - 3a_{12}a_{26})b^2}{24a_{11}}y - \frac{a_{12}}{2}xy^2 + \frac{(-2a_{12}a_{16} + a_{11}a_{26})}{6a_{11}}y^3 + \frac{a_{11}}{6}x^3\right] - cx + e \tag{12}$$

Applying $v = 0$ and $\frac{dv}{dx} = 0$ boundary conditions at point $x = l, y = 0$ (Fig. 1) in eq.(12), c and e values are determined. In this respect the deflection in y direction is

$$v = \frac{P}{I}\{\frac{(2a_{12}a_{16} - 3a_{11}a_{26})b^2}{24a_{11}}y - \frac{a_{12}}{2}xy^2 + \frac{(-2a_{12}a_{16} + a_{11}a_{26})}{6a_{11}}y^3 + \frac{a_{11}}{6}(x^3 - 3l^2x + 2l^3)\} \tag{13}$$

In order to determine deflection equation in symmetry axis of the beam, y=0 is substituted into equation (13) and then simplifying it;

$$v = \frac{P}{I}\frac{a_{11}}{6}(x^3 - 3l^2x + 2l^3) \tag{14}$$

is found.

2.2. Deflection of The Orthotropic Cantilever Subjected to Distributed Load

The stress components of the orthotropic cantilever beam subjected to distributed load are given as follows [5];

$$\sigma_x = -\frac{qx^2y}{2I} + \frac{q}{h}\left[\frac{a_{16}}{a_{11}}\frac{x}{b}\left(1 - \frac{12y^2}{b^2}\right) + 2\left(\frac{2a_{12} + a_{66}}{4a_{11}} - \frac{a_{16}^2}{a_{11}^2}\right)\left(\frac{4y^3}{b^3} - \frac{4y}{5b}\right)\right] \tag{15a}$$

$$\sigma_y = \frac{q}{2h}\left(-1+\frac{3y}{b}-\frac{4y^3}{b^3}\right) \tag{15b}$$

$$\tau_{xy} = -\frac{qx}{2I}\left(\frac{b^2}{4}-y^2\right)-\frac{qa_{16}}{ha_{11}}\left(\frac{y}{b}-\frac{4y^3}{b^3}\right) \tag{15c}$$

The procedure in distrubuted load is similar to single load procedure, if Eqs. of (15) are substituted into eqs(1). and these eqs. Are equalized to eq(4) and integrated as a function of *x* and *y*. The deflection function of *x* and *y*, respectively, are found as;

$$\begin{aligned} u = &-\frac{a_{12}qx}{2h}-\frac{a_{16}qx^2}{4bh}+\frac{9a_{12}qxy}{10bh}+\frac{a_{16}^2qxy}{5a_{11}bh}-\frac{3a_{66}qxy}{10bh}-\frac{2a_{11}qx^3y}{b^3h}-\frac{3a_{16}qx^2y^2}{b^3h} \\ &+\frac{2a_{12}qxy^3}{b^3h}-\frac{4a_{16}^2qxy^3}{a_{11}b^3h}+\frac{2a_{66}qxy^3}{b^3h}+g(y) \end{aligned} \tag{16}$$

and

$$\begin{aligned} v = &-\frac{a_{22}qy}{2h}+\frac{a_{12}a_{16}qxy}{a_{11}bh}-\frac{3a_{26}qxy}{2bh}-\frac{3a_{12}^2qy^2}{10a_{11}bh}+\frac{3a_{12}a_{16}^2qy^2}{5a_{11}^2bh}+\frac{3a_{22}qy^2}{4bh} \\ &-\frac{a_{16}a_{26}qy^2}{2a_{11}bh}-\frac{3a_{12}a_{66}qy^2}{20a_{11}bh}-\frac{3a_{12}qx^2y^2}{b^3h}-\frac{4a_{12}a_{16}qxy^3}{a_{11}b^3h}+\frac{2a_{26}qxy^3}{b^3h} \\ &+\frac{a_{12}^2qy^4}{a_{11}^2b^3h}-\frac{a_{22}qy^4}{2b^3h}-\frac{2a_{12}a_{16}^2qy^4}{a_{11}^2b^3h}-+\frac{a_{12}a_{26}qy^4}{a_{11}b^3h}+\frac{a_{12}a_{66}qy^4}{2a_{11}b^3h}+f(x) \end{aligned} \tag{17}$$

if γ_{xy} in Eq.(1.c) is equalized to γ_{xy} in Eq.(4.c), then following equation is found;

$$\begin{aligned} \gamma_{xy1}-\gamma_{xy2} = &\frac{a_{26}q}{2h}+\frac{9a_{12}qx}{10bh}-\frac{4a_{16}^2qx}{5a_{11}bh}+\frac{6a_{66}qx}{5bh}-\frac{2a_{11}qx^3}{b^3h}+\frac{8a_{12}a_{16}qy}{5a_{11}bh} \\ &-\frac{6a_{16}^3qy}{5a_{11}^2bh}-\frac{3a_{26}qy}{bh}-\frac{13a_{16}a_{66}qy}{10a_{11}bh}-\frac{8a_{12}a_{16}qy^3}{a_{11}b^3h}+\frac{8a_{16}^3qy^3}{a_{11}^2b^3h} \\ &+\frac{4a_{26}qy^3}{b^3h}-\frac{6a_{16}a_{66}qy^3}{a_{11}b^3h}-f'(x)-g'(y)=0 \end{aligned} \tag{18}$$

Because of equality of eq.(18) to zero, the summution of the terms depending *x* and the summution of the terms depending *y* should be equal to separate constants. If c is constant then;

$$A(x)-f'(x)=c \tag{19}$$

Where $A(x)=\frac{9a_{12}qx}{10bh}-\frac{4a_{16}^2qx}{5a_{11}bh}+\frac{6a_{66}qx}{5bh}-\frac{2a_{11}qx^3}{b^3h}$ from the equation (19);

$$f(x) = \int [A(x) - c]dx \tag{20}$$

Integrating this eq.

$$f(x) = \frac{(9a_{11}a_{12} - 8a_{16}^2 + 12a_{11}a_{66})qx^2}{20a_{11}bh} + \frac{a_{11}qx^4}{2b^3h} - cx + e \tag{21}$$

is found.

Substituting *f(x)* value into eq.(17) and applying $v = 0$ and $\frac{dv}{dx} = 0$ boundary conditions at point $x = l$, $y = 0$ (Fig. 1), the deflection function is found as follows;

$$\begin{aligned} v = &-\frac{9a_{12}l^2q}{20bh} + \frac{2a_{16}^2l^2q}{5a_{11}bh} - \frac{3a_{66}l^2q}{5bh} + \frac{3a_{11}ql^4}{2b^3h} + \frac{9a_{12}lqx}{10bh} - \frac{4a_{16}^2lqx}{5a_{11}bh} + \frac{6a_{66}lqx}{5bh} \\ &- \frac{2a_{11}l^3qx}{b^3h} - \frac{9a_{12}qx^2}{20bh} + \frac{2a_{16}^2qx^2}{5a_{11}bh} - \frac{3a_{66}qx^2}{5bh} + \frac{a_{11}qx^4}{2b^3h} - \frac{a_{22}qy}{2h} + \frac{a_{12}a_{16}qxy}{a_{11}bh} \\ &- \frac{3a_{26}qxy}{2bh} - \frac{3a_{12}^2qy^2}{10a_{11}bh} + \frac{3a_{12}a_{16}^2qy^2}{5a_{11}^2bh} + \frac{3a_{22}qy^2}{4bh} - \frac{a_{16}a_{26}qy^2}{2a_{11}bh} - \frac{3a_{12}a_{66}qy^2}{20a_{11}bh} \\ &- \frac{3a_{12}qx^2y^2}{b^3h} - \frac{4a_{12}a_{16}qxy^3}{a_{11}b^3h} + \frac{2a_{26}qxy^3}{b^3h} + \frac{a_{12}^2qy^4}{b^3h} - \frac{2a_{12}a_{16}^2qy^4}{a_{11}^2b^3h} \\ &- \frac{a_{22}qy^4}{b^3h} + \frac{a_{16}a_{26}qy^4}{a_{11}b^3h} + \frac{a_{12}a_{66}qy^4}{2a_{11}b^3h} \end{aligned} \tag{22}$$

In order to determine deflection equation in symmetry axis of the beam, y=0 is substituted into equation (22) and then simplifying it;

$$v = \frac{3q}{20bh}\left(-3a_{12} + \frac{8a_{16}^2}{3a_{11}} - 4a_{66}\right)(x - l)^2 + \frac{a_{11}q(3l^4 - 4l^3x + x^4)}{2b^3h} \tag{23}$$

is found.

3. AN EXAMPLE

In order to obtain the values of deflections at the free ends of the beams for different fiber direction (e.g. y=0), Eq.(14) for single load and Eq. (23) for distributed load are used. The values L, b, h, q, and P are taken as 150 mm, 40 mm, 1 mm, 2N/mm and 300 N respectively (Fig. 1). Calculations are performed for 0°, 15°, 30°, 45°, 60°, 75° and 90° fiber directions. T300/976 Grafite-Epoxy material is used for numerical calculations. The properties of the material [7] and the values of the deflections are given in Tablo 1 and in Tablo 2, respectively.

Tablo 1 The material properties of T300/976 Grafite-Epoxy

	E_1 (GPa)	E_2 (GPa)	G_{12} (GPa)	υ_{12}
T300/976 Grafite-Epoxy	156	13	7	0.23

Tablo 2. Deflections of beams at the free end *(x=0, y=0)*

θ(°)	0	15	30	45	60	75	90
Single. Load (mm)	0.405	0.928	2.192	3.531	4.423	4.792	4.867
Distri. Load (mm)	0.246	0.345	0.652	1.090	1.510	1.796	1.899

CONCLUSIONS

It can be seen from both eq.(14) and eq.(23), taking the elasticity modulus in fiber direction or orthogonal to fiber direction equal to elasticity modulus of isotropic material (i.e. E_1=E or E_2=E), in case of point load acting at the tip of the orthotropic cantilever beam, the deflection value found (Eq. 14) for the tip point (i.e. *x=0, y=0*) is the same as the deflection takes place for isotropic material under the same loading. However, in case of uniformly distributed loading condition, deflection value (Eq.23) at the tip point is different (Tablo 3).

Tablo 3. Comparison of orthotropic and isotropic beams deflections(E_1=E or E_2=E)

	Orthotropic Beam	Isotropic Beam
Point Load Case (Eq.14)	$v=\dfrac{Pl^3}{3E_1I}$	$v=\dfrac{Pl^3}{3EI}$
Distributed load case (Eq.23)	$v=\dfrac{3q}{20bh}(-3a_{12}-4a_{66})(-l)^2+\dfrac{ql^4}{8E_1I}$	$v=\dfrac{ql^4}{8EI}$

The free end deflection of the beam increases for angles ranging from 0° to 90° for both load cases due to decreasing of stiffness (Tablo 2).

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