DETERMINATION OF STRESS FUNCTIONS OF A CURVED BEAM SUBJECTED TO AN ARBITRARILY DIRECTED SINGLE FORCE AT THE FREE END

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Abstract- In this study, closed form stress functions of a curved beam are determined by theory of elasticity. The beam is subjected to a single force arbitrarily directed at the free end. Stresses are plotted for various sections of the beam and various directions of the force.

Key words- curved beam, stress function, elasticity

1. INTRODUCTION

One of the widely used members in frame structures is beams. Many investigators have studied on beams recently. Tutuncu [1] has found stress components and deflections of an orthotropic curved beam subjected to pure moment and shear load. The equations that was found have been applied to rings which can be modeled as a curved beam. Karakuzu et al. [2] have investigated elasto-plastic stress analysis in a composite beam loaded uniformly or by a single force at the free end by using an analytical solution. Özcan [3] has investigated elasto-plastic stress analysis in steel fiber reinforced thermoplastic orthotropic cantilever beam subjected to single force at the free end of the beam. Ever et al. [4] have obtained shear correction factor and deflection of a composite beam having I cross section.

In this study, the stress components of a curved beam having a rectangular cross section are determined by theory of elasticity. The beam is subjected to a single force which is arbitrarily directed at free end. The stress distributions are plotted for different sections and force directions.

2. DETERMINATION OF STRESS FUNCTIONS

Stress components in polar coordinates in theory of elasticity are given as [5];

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta}$$ (1.a)

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}$$ (1.b)

$$\tau_{r\theta} = -\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \phi}{\partial \theta})$$ (1.c)

The bending moment at any cross section of the curved beam is proportional to $Sin\theta$ and $Cos\theta$ due to $P.Cos\alpha$ and $P.Sin\alpha$ respectively (Figure 1). Therefore, solution of the problem is obtained with the use of a stress function of the type

$$\phi = \zeta_1(r)\cos\theta + \zeta_2(r)\sin\theta \tag{2}$$

where $\zeta_1(r)$ and $\zeta_2(r)$ are given in [5].

$$\zeta_1(r) = Ar^3 + \frac{B}{r} + Cr + Dr\ln r \tag{3a}$$

$$\zeta_2(r) = Er^3 + \frac{F}{r} + Gr + Hr\ln r \tag{3b}$$

where A, B,...,H are arbitrary constants found from the boundary conditions. As a result, the stress function is;

$$\phi = (Ar^3 + \frac{B}{r} + Cr + Dr\ln r)\cos\theta + (Er^3 + \frac{F}{r} + Gr + Hr\ln r)\sin\theta \tag{4}$$
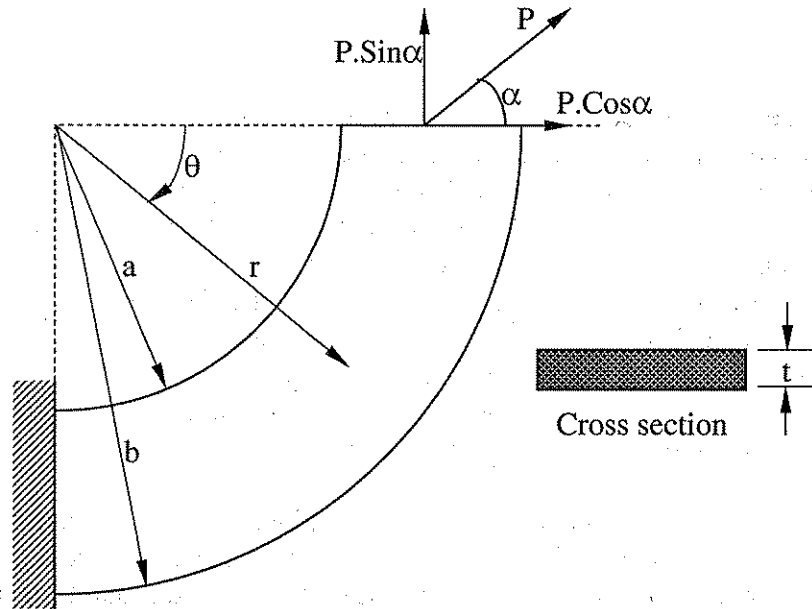

Figure 1. Illustration of the curved beam

If equation (4) is substituted into Eq.(1a, 1b and 1c), the following stresses are obtained;

$$\sigma_r = (2Ar - \frac{2B}{r^3} + \frac{D}{r})\cos\theta + (2Er - \frac{2F}{r^3} + \frac{H}{r})Sin\theta \tag{5}$$

$$\sigma_\theta = (6Ar + \frac{2B}{r^3} + \frac{D}{r})\cos\theta + (6Er + \frac{2F}{r^3} + \frac{H}{r})Sin\theta \tag{6}$$

$$\tau_{r\theta} = (2Ar - \frac{2B}{r^3} + \frac{D}{r})Sin\theta + (-2Er + \frac{2F}{r^3} + \frac{H}{r})\cos\theta \qquad (7)$$

In the above stress components, the eight coefficients are unknown. In order to find, we have to have eight boundary conditions. These are

1. $\sigma_{r=0}$ at $r=a$,

2. $\sigma_{r=0}$ at $r=b$,

3. $\tau_{r\theta} = 0$ at $r=a$,

4. $\tau_{r\theta} = 0$ at $r=b$,

5. $\int_a^b \sigma_\theta t dr = P\sin\alpha$ at $\theta = 0^o$

6. $\int_a^b \sigma_\theta t r dr = 0$ at $\theta = 0^o$

7. $\int_a^b \tau_{r\theta} t dr = P\cos\alpha$ at $\theta = 0^o$

8. $\int_a^b \tau_{r\theta} t dr = P\sin\alpha$ at $\theta = 90^o$

When these boundary conditions are applied to equations (5), (6) and (7), the coefficients are found as follows;

$$A = \frac{P\sin\alpha}{2t[-a^2 + b^2 - a^2 \ln(\frac{b}{a}) - b^2 \ln(\frac{b}{a})]} \qquad (8)$$

$$B = \frac{a^2 b^2 P\sin\alpha}{2t[a^2 - b^2 + a^2 \ln(\frac{b}{a}) + b^2 \ln(\frac{b}{a})]} \qquad (9)$$

$$C = 0 \qquad (10)$$

$$D = \frac{(a^2 + b^2)P\sin\alpha}{t[a^2 - b^2 + a^2 \ln(\frac{b}{a}) + b^2 \ln(\frac{b}{a})]} \qquad (11)$$

$$E = \frac{P \cos \alpha}{2t[-a^2 + b^2 - a^2 \ln(\frac{b}{a}) - b^2 \ln(\frac{b}{a})]} \tag{12}$$

$$F = \frac{a^2 b^2 P \cos \alpha}{2t[a^2 - b^2 + a^2 \ln(\frac{b}{a}) + b^2 \ln(\frac{b}{a})]} \tag{13}$$

$$G = 0 \tag{14}$$

$$H = \frac{(a^2 + b^2) P \cos \alpha}{t[a^2 - b^2 + a^2 \ln(\frac{b}{a}) + b^2 \ln(\frac{b}{a})]} \tag{15}$$

If the coefficients found above are inserted into equations (5), (6) and (7) and rearranged, the stress components are found as follows;

$$\sigma_r = \frac{P}{btm}\left[\frac{r}{b} + k^2\left(\frac{b}{r}\right)^3 - (1+k^2)\frac{b}{r}\right]\sin(\theta + \alpha) \tag{16}$$

$$\sigma_\theta = \frac{P}{btm}\left[3\frac{r}{b} - k^2\left(\frac{b}{r}\right)^3 - (1+k^2)\frac{b}{r}\right]\sin(\theta + \alpha) \tag{17}$$

$$\tau_{r\theta} = -\frac{P}{btm}\left[\frac{r}{b} + k^2\left(\frac{b}{r}\right)^3 - (1+k^2)\frac{b}{r}\right]\cos(\theta + \alpha) \tag{18}$$

where m and k are given as follows;

$$k = \frac{a}{b}, \quad m = 1 - k^2 + (1+k^2)\ln k$$

3. DISTRIBUTION OF THE STRESSES

To show the distribution of the stress components for different b/a ratios and values of θ, α, the beam thickness, the single load and (b-a) are taken as 1 mm, 100 N and 100 mm, respectively. The stress components are plotted for b/a ratios of 1.5, 2 and 3 (Figure 2 to 7).
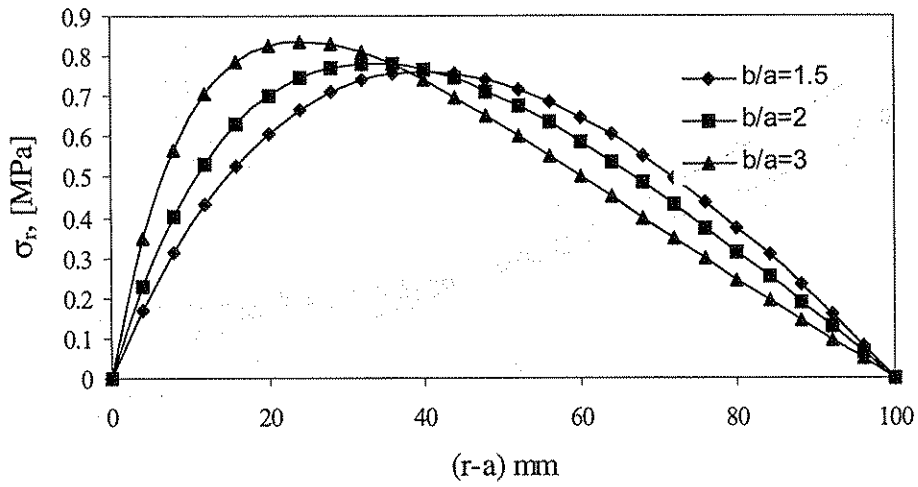
Figure 2. Distribution of the radial stress (σ_r) according to (r-a) while θ and α are 45°, 0° respectively
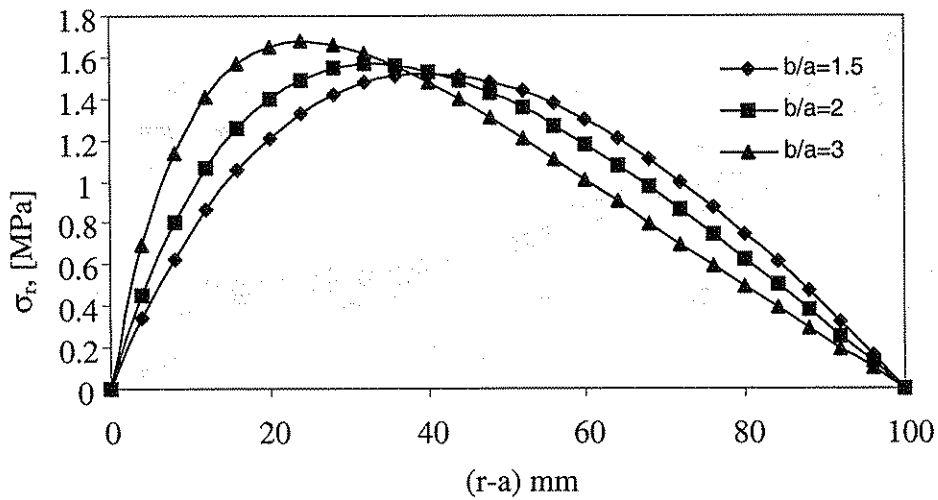

Figure 3. Distribution of the radial stress (σ_r) according to (r-a) while θ and α are 45°, 45° respectively
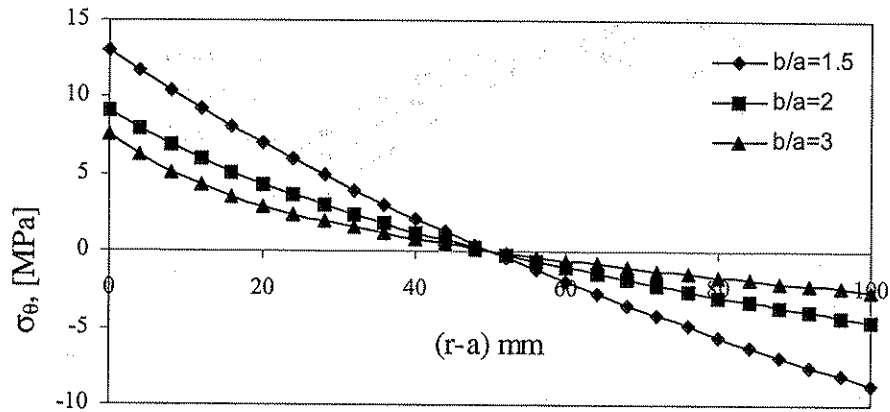
Figure 4. Distribution of the tangential stress (σ_θ) according to (r-a) while θ and α are $0°$, $45°$ respectively
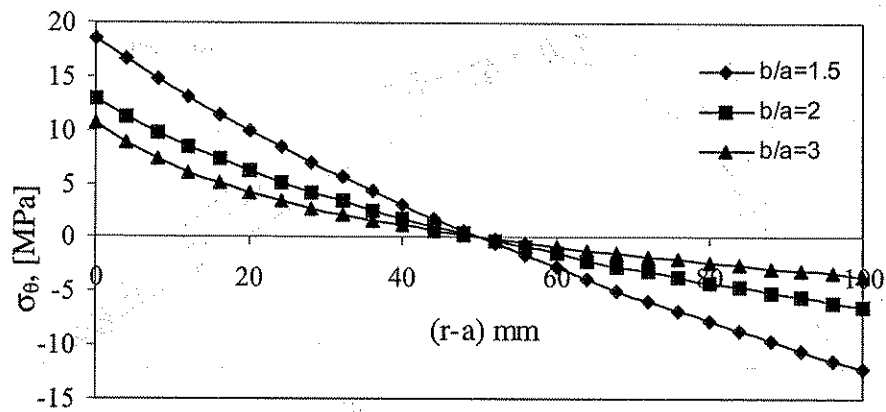

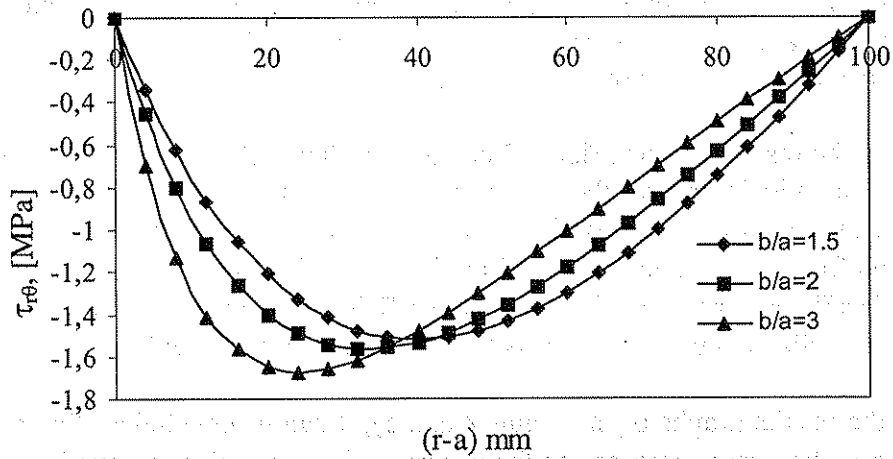Figure 5. Distribution of the tangential stress (σ_θ) according to (r-a) while θ and α are $0°$, $90°$ respectively

Figure 6. Distribution of the shear stress($\tau_{r\theta}$) according to (r-a) while θ and α are 0°, 0° respectively
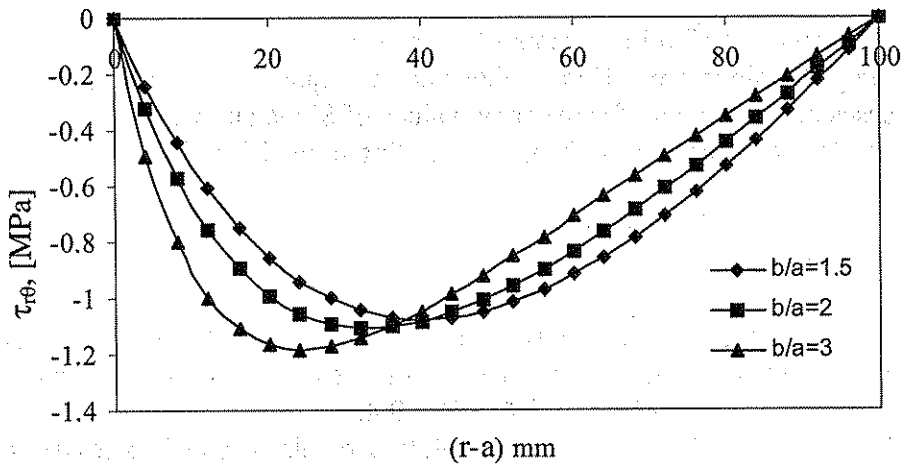

Figure 7. Distribution of the shear stress ($\tau_{r\theta}$) according to (r-a) while θ and α are 0°, 45° respectively

4. CONCLUSION

In this study, a curved beam with an arbitrarily directed end force is analysed and formulations of stress components, σ_r, $\tau_{r\theta}$ and σ_θ, are carried out. Because of the end force, eight integration constants are needed to set up expressions. Using these expressions, stress components(σ_r, $\tau_{r\theta}$ and σ_θ) versus (r-a) graphs are plotted for b/a ratios of 1.5, 2, 3 and for varying angles of θ and α. P and (b-a) are taken as 100 N and 100 mm respectively, in all these graphs.

By analyzing the results graphs (Figure 2 and 3) of radial stress (σ_r), it can be concluded that as the b/a ratio increases, the stress increases and the place of maximum stress along the beam height (b-a) approaches to inner surface. Considering equation (16) it's obvious that effect of $\theta+\alpha$ and $180-(\theta+\alpha)$ are identical. Two different combinations of θ and α (i.e, $\theta=45°,\alpha=0°$ and $\theta=45°,\alpha=45°$) are sufficient to demonstrate all combinations of θ and α for radial and tangential stress for the value of 45° and 90°.

Analyzing the results graphs of σ_θ (Figure 4 and 5), it can be concluded that as the b/a ratio increases, the stress decreases on the contrary of σ_r. Considering equation (17), it is obvious that the maximum stress occurs when the total of θ, α is 90°. For the same reason as explained above two graphs of tangential stress distribution are plotted for the pairs of $\theta=0°,\alpha=45°$ and $\theta=0°,\alpha=90°$.

It is surprising that when a horizontal cantilever beam subjected to a single axial force, there is only uniformly distributed normal stress. However, the tangential stress at the free end has a quadratical distributed form while the direction of the force is 90°(Fig.5).

It is seen that stress distribution graphs of shear stress and radial stress are similar to each other,only the signs are different. Considering equations of (16), (18), it is seen that the stresses are both equal for the same values of Sinus and Cosines. Also it can be seen that the place of neutral axis changes according to the b/a ratio.

Overviewing all stress distribution graphs it is seen that value of tangential stress is greater than the σ_r, $\tau_{r\theta}$.

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