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A FIXED POINT THEOREM ON CONE METRIC SPACES WITH NEW TYPE CONTRACTIVITY

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ABSTRACT. In the present work, a common fixed point theorem for self maps on cone metric spaces is proved. Also two examples, which shows that our main theorem is generalized version of main theorems of [A. Branciari, Int. J. Math. Math. Sci., 29 (2002), no. 9, 531–536] and [L.G. Huang and X. Zhang, J. Math. Anal. Appl. 332 (2007), no. 2, 1468–1476] are given.

1. INTRODUCTION

There are a lot of generalizations of Banach fixed point principle in the literature. One of the most interesting generalization of it has been given by Branciari [10]. Branciari has made this by taking integral type contraction instead of ordinary contraction. He has given an example showing that the integral type contraction is more general than ordinary contraction. This interesting fixed point result is as follows:

Theorem 1.1 ([10]). *Let (X, d) be a complete metric space, $k \in (0, 1)$ and $T : X \to X$ be mapping such that for each $x, y \in X$ one has*

$$\int_0^{d(Tx,Ty)} f(t)dt \leq k \int_0^{d(x,y)} f(t)dt \tag{1.1}$$

where $f : [0, \infty) \to [0, \infty]$ is a Lebesque integrable mapping which is finite integral on each compact subset of $[0, \infty)$, non-negative and such that for each $t > 0$,

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$\int_0^t f(s)ds > 0$. *Then T has a unique fixed point $z \in X$ such that for each $x \in X$, $\lim_{n\to\infty} T^n x = z$.*

Since then many fixed point theorists have given some fixed point and common fixed point results using integral type contractions (see for example [3, 4, 5, 18, 23, 24, 27]).

On the other hand, existence of fixed point in cone metric space has been considered recently in [14]. Before giving this result, we recall some definitions and properties of cone metric spaces.

Let E be a real Banach space and P be a subset of E. By θ we denote the zero element of E and by $IntP$ the interior of P. The subset P is called a cone if and only if:

(i) P is closed, nonempty and $P \neq \{\theta\}$,
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Longrightarrow ax + by \in P$,
(iii) $x \in P$ and $-x \in P \Longrightarrow x = \theta$.

A cone P is called solid if it contains interior points, that is, if $IntP \neq \emptyset$. Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$, we shall write $x \ll y$ if $y - x \in IntP$. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies $\|x\| \leq K \|y\|$. The least positive number satisfying the above is called the normal constant of P.

The following example shows that there are non-normal cones.

Example 1.2 ([22])**.** Let $E = C_R^2([0,1])$ with the norm $||f|| = ||f||_\infty + ||f'||_\infty$ and consider the cone $P = \{f \in E : f \geq 0\}$. For each $K > 1$, put $f(x) = x$ and $g(x) = x^{2K}$. Then $0 \leq g \leq f$, $||f|| = 2$ and $||g|| = 2K + 1$. Since $K\,||f|| < ||g||$, K is not normal constant of P . Therefore, P is a non-normal cone.

In the following we always suppose that E is a real Banach space, P is a cone in E with $IntP \neq \emptyset$ and $\preceq$ a partial ordering with respect to P.

Definition 1.3 ([14])**.** Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(d_1) $\theta \prec d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$,
(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(d_3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

It is obvious that the cone metric spaces generalize metric spaces.

Example 1.4 ([14])**.** Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E \mid x, y \geq 0\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha |x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 1.5 ([14])**.** Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $\theta \ll c$ there is N such that for all $n > N$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. If for every $c \in E$ with $\theta \ll c$ there is N such that for all $n, m > N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X. (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Lemma 1.6 ([14]). *Let (X, d) be a cone metric space, P be a normal cone and let $\{x_n\}$ be a sequence in X. Then*

(i) *$\{x_n\}$ converges to x if and only if $d(x_n, x) \to \theta$ $(n \to \infty)$,*
(ii) *$\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$ $(n, m \to \infty)$.*

Let (X, d) be a cone metric space, $T : X \to X$ and $x_0 \in X$. Then the function T is continuous at x_0 if for any sequence $x_n \to x_0$ we have $Tx_n \to Tx_0$ [15].

Now we give the result of Huang and Zhang.

Theorem 1.7 ([14]). *Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping $f : X \to X$ satisfies the contractive condition*

$$d(fx, fy) \preceq kd(x, y) \text{ for all } x, y \in X, \tag{1.2}$$

where $k \in [0, 1)$ is a constant. Then f has a unique fixed point in X.

Also, Huang and Zhang [14] gave an example showing that Theorem 1.7 is a generalization of Banach fixed point principle.

After the definition of the concept of cone metric space, fixed point theory on these spaces has been developing (see for example [1, 2, 6, 7, 8, 9, 11, 13, 16, 17, 19, 20, 21, 22, 25, 26]).

The aim of this work is to give a fixed point result using the ideas of Branciari [10] (or Zhang [27]) and Huang and Zhang [14]. Our result a generalized version of both Theorem 1.1 and Theorem 1.7 in cone metric spaces.

Let (X, d) be a cone metric space with the real Banach space E and the cone P. Let $Q_a = \{u \in P : \|u\| < a, a \in (0, \infty]\}$ and let $F : Q_a \to P$ satisfy that

(F_1) $F(\theta) = \theta$ and $\theta \prec F(u)$ for each $u \in Q_a \backslash \{\theta\}$,
(F_2) F is nondecreasing on Q_a, that is, $u, v \in Q_a$ and $u \preceq v$ implies $F(u) \preceq F(v)$,
(F_3) F is continuous.
(F_4) $F(u_n) \to \theta$ implies $u_n \to \theta$ as $n \to \infty$.

Define $\mathcal{F}(E, P, Q_a) = \{F \mid F : Q_a \to P \text{ satisfies } (F_1)\text{-}(F_4)\}$.

Example 1.8.

(a) Let E be a real Banach space and P be a cone in E. Define $F : Q_a \to P$ with $F(u) = u$, then $F \in \mathcal{F}(E, P, Q_a)$ for all $a \in (0, \infty]$.
(b) Let $E = \mathbb{R}$ be real numbers, $P_1 = [0, \infty)$. Suppose that ϕ is nonnegative, Lebesgue integrable on $[0, \infty)$ and satisfies

$$\int_0^\varepsilon \phi(s)ds > 0 \text{ for each } \varepsilon > 0.$$

Let $F(u) = \int_0^u \phi(s)ds$. Then $F \in \mathcal{F}(\mathbb{R}, P_1, Q_a)$ for all $a \in (0, \infty]$.

(c) Let $P_2 = \{(x,y) \in \mathbb{R}^2 : x,y \geq 0\}$ and $F_1(u) = (\|u\|, \alpha\|u\|)$. Then $F_1 \in \mathcal{F}(\mathbb{R}^2, P_2, Q_a)$ for all $a \in (0,\infty]$, where $\alpha \geq 0$. Similarly, let

$$F_2(u) = \left(\int_0^{\|u\|} \phi(s)ds, \alpha \int_0^{\|u\|} \phi(s)ds \right).$$

Then $F_2 \in \mathcal{F}(\mathbb{R}^2, P_2, Q_a)$ for all $a \in (0,\infty]$, where α and ϕ as above.

(d) Let $E = C_{\mathbb{R}}[0,1]$ with the supremum norm, $P_3 = \{f \in E : f \geq 0\}$ and $F(u) = \dfrac{\|u\|}{1+\|u\|}\psi$. Then $F \in \mathcal{F}(E, P_3, Q_a)$ for all $a \in (0,\infty]$, where $\psi \in IntP_3$.

(e) Let $E = l^1$, $P_4 = \{\{x_n\} \in E : x_n \geq 0, \text{ for all } n\}$ and $F(u) = \dfrac{\|u\|}{2^n}$. Then $F \in \mathcal{F}(E, P_4, Q_a)$ for all $a \in (0,\infty]$.

Definition 1.9 ([11])**.** Let P be a cone on a Banach space E. A nondecreasing function $\varphi : P \to P$ is called φ-map if

(φ_1) $\varphi(\theta) = \theta$ and $\theta \prec \varphi(\omega) \prec \omega$ for $\omega \in P\backslash\{\theta\}$,
(φ_2) $\omega \in IntP$ implies $\omega - \varphi(\omega) \in IntP$,
(φ_3) $\lim_{n\to\infty} \varphi^n(\omega) = \theta$ for every $\omega \in P\backslash\{\theta\}$.

There is an example of φ-map in [12].

2. Main results

Let E be a real Banach space, P be a cone in E and $\preceq$ is partial ordering with respect to P. We consider the following condition for convenience:

$$\begin{cases} \text{If } \{u_n\} \text{ is a sequence in } E \text{ such that } \|u_n\| \text{ is convergent,} \\ \text{then } \{u_n\} \text{ has an upper limit in } E \text{ w.r.t } \preceq. \end{cases} \tag{2.1}$$

It is clear that $\mathbb{R}^2$ is satisfied the condition (2.1) with the cone P_2. In the following theorem we assume that E satisfies the condition (2.1).

Theorem 2.1. *Let (X,d) be a complete cone metric space with the normal cone P on a Banach space E. Let $D = \sup\{\|d(x,y)\| : x,y \in X\}$ and set $a = D$ if $D = \infty$ and $a > D$ if $D < \infty$. Suppose that $T,S : X \to X$, $F \in \mathcal{F}(E,P,Q_a)$ satisfy*

$$F(d(Tx,Sy)) \preceq \varphi(F(d(x,y))), \tag{2.2}$$

for all $x,y \in X$, where φ is a continuous φ-map. Then T and S have a unique common fixed point in X.

Proof. First we prove that any fixed point of T is also a fixed point of S, and conversely. If $Tx = x$ but $Sx \neq x$, then we have from (2.2)

$$F(d(x,Sx)) = F(d(Tx,Sx)) \preceq \varphi(F(\theta)) = \theta,$$

which is a contradiction. So $Sx = x$. Similarly, if $Sx = x$, then $Tx = x$.

Now we show that if T and S have a common fixed point, then the fixed point is unique. Let $Tx = Sx = x$ and $Ty = Sy = y$. If $x \neq y$, then we have from (2.2)

$$F(d(x,y)) = F(d(Tx,Sy)) \preceq \varphi(F(d(x,y))),$$

which is a contradiction. Thus $x = y$.

Let $x_0 \in X$ and let $x_{2n+1} = Tx_{2n}$, $x_{2n+2} = Sx_{2n+1}$ for all $n = 0, 1, 2, \cdots$. Suppose that for any n, $x_{n+1} \neq x_n$. Otherwise T or S has a fixed point and the proof is complete. Now we show that

$$\begin{aligned} F(d(x_{n+1}, x_n)) &\preceq \varphi(F(d(x_n, x_{n-1}))) \\ &\prec F(d(x_n, x_{n-1})) \text{ for } n \geq 1. \end{aligned} \tag{2.3}$$

We have that

$$\begin{aligned} F(d(x_{2n+1}, x_{2n})) &= F(d(Tx_{2n}, Sx_{2n-1})) \\ &\preceq \varphi(F(d(x_{2n}, x_{2n-1}))) \\ &\prec F(d(x_{2n}, x_{2n-1})). \end{aligned} \tag{2.4}$$

Similarly we have

$$\begin{aligned} F(d(x_{2n+2}, x_{2n+1})) &= F(d(Sx_{2n+1}, Tx_{2n})) \\ &\preceq \varphi(F(d(x_{2n+1}, x_{2n}))) \\ &\prec F(d(x_{2n+1}, x_{2n})). \end{aligned} \tag{2.5}$$

Therefore from (2.4) and (2.5) we have (2.3). Then from (2.3)

$$\begin{aligned} F(d(x_{n+1}, x_n)) &\preceq \varphi(F(d(x_n, x_{n-1}))) \\ &\preceq \varphi^2(F(d(x_{n-1}, x_{n-2}))) \\ &\ \ \vdots \\ &\preceq \varphi^n(F(d(x_1, x_0))). \end{aligned}$$

Fix $\theta \ll c$ and we choose a positive real number δ such that $c + N_\delta(\theta) \subset IntP$, where $N_\delta(\theta) = \{x \in P : \|x\| < \delta\}$. Also, choose a natural number n_0 such that $\varphi^n(F(d(x_1, x_0))) \in N_\delta(\theta)$, for all $n \geq n_0$. Then

$$\varphi^n(F(d(x_1, x_0))) \ll c,$$

for all $n \geq n_0$. Consequently $F(d(x_{n+1}, x_n)) \ll c$, for all $n \geq n_0$. From (F_4), we have, for all $\theta \ll c$, there exists a natural number n_0 such that for all $n \geq n_0$

$$d(x_{n+1}, x_n) \ll c.$$

Now we claim that $\{x_n\}$ is a Cauchy sequence in X. If not, there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that

$$\varepsilon \leq \left\| d\left(x_{m_k}, x_{n_k}\right) \right\|.$$

Further, corresponding to odd numbers m_k, we can choose even numbers n_k in such a way that it is the smallest integer with $n_k > m_k$ such that $\left\| d\left(x_{m_k}, x_{n_{k-2}}\right) \right\| < \varepsilon$ and $\varepsilon \leq \left\| d\left(x_{m_k}, x_{n_k}\right) \right\|$. From

$$\begin{aligned} \varepsilon &\leq \left\| d\left(x_{m_k}, x_{n_k}\right) \right\| \\ &\leq \left\| d\left(x_{m_k}, x_{n_{k-2}}\right) \right\| + \left\| d(x_{n_{k-2}}, x_{n_k}) \right\| < \varepsilon + \left\| d\left(x_{n_{k-2}}, x_{n_k}\right) \right\|, \end{aligned}$$

it follows

$$\lim_{k\to\infty} \|d(x_{m_k}, x_{n_k})\| = \varepsilon.$$

Now

$$\left\|d\left(x_{n_{k-1}}, x_{m_{k-1}}\right)\right\| \leq \left\|d\left(x_{n_{k-1}}, x_{n_k}\right)\right\| + \|d(x_{n_k}, x_{m_k})\| + \left\|d\left(x_{m_k}, x_{m_{k-1}}\right)\right\|$$

and

$$\|d(x_{n_k}, x_{m_k})\| \leq \left\|d\left(x_{n_k}, x_{n_{k-1}}\right)\right\| + \left\|d(x_{n_{k-1}}, x_{m_{k-1}})\right\| + \left\|d\left(x_{m_{k-1}}, x_{m_k}\right)\right\|$$

gives

$$\lim_{k\to\infty} \left\|d\left(x_{m_{k-1}}, x_{n_{k-1}}\right)\right\| = \varepsilon.$$

Therefore from the condition (2.1) there exists $\theta \ll c$ such that

$$\limsup_{k\to\infty} d(x_{m_k}, x_{n_k}) = \limsup_{k\to\infty} d\left(x_{m_{k-1}}, x_{n_{k-1}}\right) = c.$$

Setting $x = x_{m_{k-1}}$ and $y = x_{n_{k-1}}$ in (2.2), we obtain

$$\begin{aligned} F\left(d(x_{m_k}, x_{n_k})\right) &= F(d(Tx_{m_{k-1}}, Sx_{n_{k-1}})) \\ &\preceq \varphi(F(d(x_{m_{k-1}}, x_{n_{k-1}}))) \end{aligned}$$

which on take upper limit

$$F(c) \preceq \varphi(F(c)) \prec F(c)$$

a contradiction which shows that $\{x_n\}$ is a Cauchy sequence in X, and so $x_n \to u$, for some u in X. Take $x = x_{2n}$ and $y = u$ in (2.2), we have

$$F(d(Tx_{2n}, Su)) \preceq \varphi(F(d(x_{2n}, u)))$$

and

$$F(d(u, Su)) = \varphi(F(\theta)) = \theta$$

implies that $d(u, Su) = \theta$, and $u = Su$. □

Corollary 2.2. *Let (X, d) be a complete cone metric space with the normal cone P on a Banach space E. Let $D = \sup\{\|d(x, y)\| : x, y \in X\}$ and set $a = D$ if $D = \infty$ and $a > D$ if $D < \infty$. Suppose that $T : X \to X$, $F \in \mathcal{F}(E, P, Q_a)$ satisfy*

$$F(d(Tx, Ty)) \preceq \varphi(F(d(x, y))), \tag{2.6}$$

for all $x, y \in X$, where φ is a continuous φ-map. Then T has a unique fixed point in X.

Remark 2.3. Corollary 2.2 is a generalized version of Theorem 1.1. Indeed, if we take $E = \mathbb{R}$, $P_1 = [0, \infty)$, $F \in \mathcal{F}(\mathbb{R}, P_1, Q_a)$ as in Example 1.8 (b) and $\varphi(\omega) = k\omega$ ($k \in [0, 1)$) in Corollary 2.2, we have Theorem 1.1.

The following example supports this remark.

Example 2.4. Let $E = \mathbb{R}^2$ with the Euclidean norm, and $P_2 = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ a cone in E. Let

$$X = \{(x, 0) \in \mathbb{R}^2 : 0 \leq x \leq 1\} \cup \{(0, x) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$$

and let $d : X \times X \to E$ defined by (as in [14])

$$\begin{aligned} d((x,0),(y,0)) &= (\frac{4}{3}|x-y|, |x-y|), \\ d((0,x),(0,y)) &= (|x-y|, \frac{2}{3}|x-y|) \\ d((x,0),(0,y)) &= d((0,y),(x,0)) = (\frac{4}{3}x+y, x+\frac{2}{3}y). \end{aligned}$$

Then (X,d) is a complete cone metric space.

Let mapping $T : X \to X$ with $T(x,0) = (0,x)$ and $T(0,x) = (\frac{x}{2},0)$. It is easy to see that T satisfies the condition (2.6) of Corollary 2.2 with $F(u) = u$ and $\varphi(\omega) = \frac{3}{4}\omega$. Therefore, we can apply Corollary 2.2 to this example. Note that, the conditions of Theorem 1.1 are not satisfied if d is the usual metric in $\mathbb{R}^2$. For example, if d is the usual metric in $\mathbb{R}^2$, then there are not a function f and a constant $k \in (0,1)$ satisfying the condition (1.1) for $x = (1,0)$ and $y = (0,0)$ in X.

Remark 2.5. Theorem 2.1 is a generalized version of Theorem 1.7. Indeed, if we take $F \in \mathcal{F}(E,P,Q_a)$ defined by $F(u) = u$ and $\varphi(\omega) = k\omega$ ($k \in [0,1)$) in Theorem 2.1, we have Theorem 1.7.

The following example supports this remark.

Example 2.6. Let $E = \mathbb{R}^2$ with the maximum norm, and $P_2 = \{(x,y) \in \mathbb{R}^2 : x,y \geq 0\}$ a cone in E. Let $X = \{\frac{1}{n} : n = 1,2,\cdots\} \cup \{0\}$ and let $d : X \times X \to E$ defined by $d(x,y) = (|x-y|, |x-y|)$. Then (X,d) is a complete cone metric space and $D = 1$. Let $F(\theta) = \theta$ and $F(u) = \left(\|u\|^{\frac{1}{\|u\|}}, \alpha \|u\|^{\frac{1}{\|u\|}}\right)$, for $u \in Q_e \backslash \{\theta\}$. Then $F \in \mathcal{F}(\mathbb{R}^2, P_2, Q_e)$ and let $\varphi(\omega) = \frac{1}{2}\omega$. Suppose that T and S are mappings on X defined by

$$Tx = Sx = \begin{cases} 0 & , \quad x = 0 \\ \dfrac{1}{n+1} & , \quad x = \dfrac{1}{n}. \end{cases}$$

Now we prove that, for each $x,y \in X$,

$$F(d(Tx,Sy)) \preceq \varphi(F(d(x,y))).$$

For each $x,y \in X$ we have

$$\begin{aligned} F(d(Tx,Sy)) &= \left(\|d(Tx,Sy)\|^{\dfrac{1}{\|d(Tx,Sy)\|}}, \alpha \|d(Tx,Sy)\|^{\dfrac{1}{\|d(Tx,Sy)\|}} \right) \\ &= \left(|Tx-Sy|^{\dfrac{1}{|Tx-Sy|}}, \alpha |Tx-Sy|^{\dfrac{1}{|Tx-Sy|}} \right), \end{aligned}$$

since $\|d(Tx,Sy)\| = \|(|Tx-Sy|, |Tx-Sy|)\| = |Tx-Sy|$. On the other hand, for each $x,y \in X$ we have

$$\begin{aligned} \varphi(F(d(x,y))) &= \frac{1}{2}F(d(x,y)) \\ &= \frac{1}{2}\left(\|d(x,y)\|^{\frac{1}{\|d(x,y)\|}}, \alpha \|d(x,y)\|^{\frac{1}{\|d(x,y)\|}} \right) \\ &= \frac{1}{2}\left(|x-y|^{\frac{1}{|x-y|}}, \alpha |x-y|^{\frac{1}{|x-y|}} \right). \end{aligned}$$

Now using Example 2 of [27] for each $x,y \in X$ we have

$$|Tx-Sy|^{\frac{1}{|Tx-Sy|}} \leq \frac{1}{2}\left(|x-y|^{\frac{1}{|x-y|}} \right)$$

and therefore we have

$$\begin{aligned} F(d(Tx,Sy)) &= \left(|Tx-Sy|^{\frac{1}{|Tx-Sy|}}, \alpha |Tx-Sy|^{\frac{1}{|Tx-Sy|}} \right) \\ &\preceq \frac{1}{2}\left(|x-y|^{\frac{1}{|x-y|}}, \alpha |x-y|^{\frac{1}{|x-y|}} \right) \\ &= \varphi(F(d(x,y))). \end{aligned}$$

This shows that the condition (2.2) of Theorem 2.1 is satisfied. Now suppose there exists $k \in [0,1)$ such that, for all $x,y \in X$,

$$d(Tx,Sy) \preceq kd(x,y).$$

This implies, for all $x,y \in X$,

$$(|Tx-Ty|, |Tx-Ty|) \preceq k(|x-y|, |x-y|)$$

or

$$|Tx-Ty| \leq k|x-y|. \tag{2.7}$$

But we can not get the inequality (2.7). To see this, let $x = \frac{1}{n}$ and $y = \frac{1}{n+1}$. Then we have

$$|Tx-Sy| = \frac{1}{(n+1)(n+2)}, \qquad |x-y| = \frac{1}{n(n+1)}$$

and so

$$\sup_{x,y\in X, x\neq y} \frac{|Tx-Sy|}{|x-y|} \geq \sup_{n\in\mathbb{N}} \frac{\left|T\frac{1}{n} - S\frac{1}{n+1}\right|}{\left|\frac{1}{n} - \frac{1}{n+1}\right|} = 1.$$

Therefore the condition (1.2) of Theorem 1.7 is not satisfied.

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