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Approximation by a complex q-Baskakov-Stancu operator in compact disks

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Abstract
In this paper, we consider a complex q-Baskakov-Stancu operator and study some approximation properties. We give a quantitative estimate of the convergence, Voronovskaja-type result and exact order of approximation in compact disks.

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1 Introduction

Recently complex approximation operators have been studied intensively. For this approach, we refer to the book of Gal [1], where he considers approximation properties of several complex operators such as Bernstein, q-Bernstein, Favard-Szasz-Mirakjan, Baskakov and some others. Also we refer to the useful book of Aral, Gupta and Agarwal [2] who consider many applications of q-calculus in approximation theory. Now, for the construction of the new operators, we give some notations on q-analysis [3, 4].

Let $q > 0$. The q-integer $[n]$ and the q-factorial $[n]!$ are defined by

$$[n] := [n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1, \\ n, & q = 1 \end{cases} \quad \text{for } n \in \mathbb{N}$$

and

$$[n]! := \begin{cases} [1]_q[2]_q \cdots [n]_q, & n = 1, 2, \dots, \\ 1, & n = 0 \end{cases} \quad \text{for } n \in \mathbb{N} \text{ and } [0]! = 1,$$

respectively. For integers $n \geq r \geq 0$, the q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q![n-r]_q!}.$$

The q-derivative of $f(z)$ is denoted by $D_q f(z)$ and defined as

$$D_q f(z) := \frac{f(qz) - f(z)}{(q-1)z}, \quad z \neq 0, \qquad D_q f(0) = f'(0),$$

also

$$D_q^0 f := f, \qquad D_q^n f := D_q\bigl(D_q^{n-1} f\bigr), \quad n = 1, 2, \ldots$$

q-Pochhammer formula is given by

$$(x,q)_0 = 1,$$
$$(x,q)_n = \prod_{k=0}^{n-1}\bigl(1 - q^k x\bigr)$$

with $x \in \mathbb{R}$, $n \in \mathbb{N} \cup \{\infty\}$. The q-derivative of the product and the quotient of two functions f and g are

$$D_q\bigl(f(z)g(z)\bigr) = f(z)D_q\bigl(g(z)\bigr) + g(qz)D_q\bigl(f(z)\bigr)$$

and

$$D_q\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)D_q(f(z)) - f(z)D_q(g(z))}{g(z)g(qz)},$$

respectively (see in [3]). Moreover, we have

$$[x_0, x_1, \ldots, x_m; f \cdot g] = \sum_{i=0}^{m} [x_0, x_1, \ldots, x_i; f][x_i, x_{i+1}, \ldots, x_m; g], \tag{1.1}$$

where $[x_0, x_1, \ldots, x_m; f]$ denotes the divided difference of the function f on the knots $x_0, x_1, \ldots, x_m$ (see [4] also [5]).

In [6], Aral and Gupta constructed the q-Baskakov operator as

$$Z_n^q(f)(x) = \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} q^{\frac{k(k-1)}{2}} z^k (-x,q)_{n+k}^{-1} f\left(\frac{[k]}{q^{k-1}[n]}\right), \quad n \in \mathbb{N},$$

where $x \geq 0$, $q > 0$ and f is a real-valued continuous function on $[0,\infty)$. The authors studied the rate of convergence in a polynomial weighted norm and gave a theorem related to monotonic convergence of the sequence of operators with respect to n. Not only they proved a kind of monotonicity by means of q-derivative but also they expressed the operator in terms of divided differences as follows:

$$W_{n,q}(f)(x) = \sum_{j=0}^{\infty} \frac{[n+j-1]!}{[n-1]!} q^{\frac{-j(j-1)}{2}} \left[0, \frac{1}{[n]}, \frac{[2]}{q[n]}, \ldots, \frac{[j]}{q^{j-1}[n]}; f\right] \frac{x^j}{[n]^j} \tag{1.2}$$

$n \in \mathbb{N}$, similar to the case of classical Baskakov operators in the sense of Lupaş in [7]. That is to say, $Z_n^q(f)(x) = W_{n,q}(f)(x)$ for $x \geq 0$ and $q > 0$, so they proved that

$$\left[0, \frac{1}{[n]}, \frac{[2]}{q[n]}, \ldots, \frac{[j]}{q^{j-1}[n]}; f\right] = \frac{q^{j(j-1)} \nabla_q^j f(0)}{[j]!}[n]^j = \frac{f^{(j)}(\zeta)}{j!}, \quad \zeta \in \left(0, \frac{[j]}{q^{j-1}[n]}\right), \tag{1.3}$$

where ∇_q^r stands for q-divided differences given by $\nabla_q^0 f(x_j)$,

$$\nabla_q^{r+1} f(x_j) = q^r \nabla_q^r f(x_{j+1}) - \nabla_q^r f(x_j)$$

for $r \in \mathbb{N} \cup \{0\}$.

A different type of the q-Baskakov operator was also given by Aral and Gupta in [8]. In [9] Finta and Gupta studied the q-Baskakov operator $Z_n^q(f)(x)$ for $0 < q < 1$. Using the second-order Ditzian-Totik modulus of smoothness, they gave direct estimates. They also introduced the limit q-Baskakov operator.

In [10] Gupta and Radu introduced a q-analogue of Baskakov-Kantorovich operators and studied weighted statistical approximation properties of them for $0 < q < 1$. They also obtained some direct estimations for error with the help of weighted modulus of smoothness. Moreover, Durrmeyer-type modifications of q-Baskakov operators were studied in [11] and [12]. In [13], Söylemez, Tunca and Aral defined a complex form of q-Baskakov operators by

$$W_{n,q}(f)(z) = \sum_{j=0}^{\infty} \frac{[n+j-1]!}{[n-1]!} q^{\frac{-j(j-1)}{2}} \left[0, \frac{1}{[n]}, \frac{[2]}{q[n]}, \dots, \frac{[j]}{q^{j-1}[n]}; f\right] \frac{z^j}{[n]^j} \tag{1.4}$$

for $q > 1, f : \overline{D}_R \cup [R, \infty) \to \mathbb{C}$, replacing x by z in the operator $W_{n,q}(f)(x)$ given by (1.2). They obtained a quantitative estimate for simultaneous approximation, Voronovskaja-type result and degree of simultaneous approximation in compact disks.

In recent years, a Stancu-type generalization of the operators has been studied. Büyükyazıcı and Atakut considered a Stancu-type generalization of the real Baskakov operators in [14]. Also in [15], q-Baskakov-Beta-Stancu operators were introduced. In [16] Gupta-Verma studied the Stancu-type generalization of complex Favard-Szasz-Mirakjan operators and established some approximation results in the complex domain. In [17] Gal, Gupta, Verma and Agrawal introduced complex Baskakov-Stancu operators and studied Voronovskaja-type results with quantitative estimates for these operators attached to analytic functions on compact disks.

Now we define a new type of the complex q-Baskakov-Stancu operator

$$\begin{aligned} W_{n,q}^{\alpha,\beta}(f)(z) = {} & \sum_{j=0}^{\infty} \frac{[n+j-1]!}{[n-1]!} q^{\frac{-j(j-1)}{2}} \\ & \times \left[\frac{[\alpha]}{[n]+[\beta]}, \frac{[\alpha]+[1]}{[n]+[\beta]}, \dots, \frac{q^{j-1}[\alpha]+[j]}{q^{j-1}([n]+[\beta])}; f\right] \frac{z^j}{([n]+[\beta])^j}, \end{aligned} \tag{1.5}$$

where $0 \le \alpha \le \beta$; for $j = 0$, we take $[n][n+1]\cdots[n+j-1] = 1$. We suppose that f is analytic on the disk $|z| < R$, $R > 1$ and has exponential growth in the compact disk with all derivatives bounded in $[0, \infty)$ by the same constant.

Note that taking $\alpha = \beta = 0$, $W_{n,q}^{\alpha,\beta}(f)(z)$ reduces to the complex q-Baskakov operator $W_{n,q}(f)(z)$ given in (1.4).

In this work, for such f and $q > 1$, we study some approximation properties of the complex q-Baskakov-Stancu operator which is defined by forward differences.

2 Auxiliary results

In this section, we give some results which we shall use in the proof of theorems.

Lemma 1 *Let us define* $e_k(z) = z^k$, $T_{n,k}^{\alpha,\beta}(z) := W_{n,q}^{\alpha,\beta}(e_k)(z)$, *and* $\mathbb{N}^0$ *denotes the set of all non-negative integers. Then, for all* $n, k \in \mathbb{N}^0$, $0 \leq \alpha \leq \beta$ *and* $z \in \mathbb{C}$, *we have the following recurrence formula*:

$$T_{n,k+1}^{\alpha,\beta}(z) = \frac{qz(1+\frac{z}{q})}{[n]+[\beta]} D_q T_{n,k}^{\alpha,\beta}\left(\frac{z}{q}\right) + \frac{[n]z+[\alpha]}{([n]+[\beta])} T_{n,k}^{\alpha,\beta}(z). \tag{2.1}$$

Hence

$$T_{n,1}^{\alpha,\beta}(z) = \frac{[n]z+[\alpha]}{[n]+[\beta]}, \qquad T_{n,2}^{\alpha,\beta}(z) = \frac{z(1+\frac{z}{q})}{[n]+[\beta]} \frac{[n]}{[n]+[\beta]} + \left(\frac{[n]z+[\alpha]}{[n]+[\beta]}\right)^2$$

for all $z \in \mathbb{C}$.

Proof Now we can write

$$\begin{aligned} T_{n,k}^{\alpha,\beta}(z) = {} & \sum_{j=0}^{\infty} \frac{[n][n+1]\cdots[n+j-1]}{([n]+[\beta])^j} q^{\frac{-j(j-1)}{2}} \\ & \times \left[\frac{[\alpha]}{([n]+[\beta])}, \ldots, \frac{q^{j-1}[\alpha]+[j]}{q^{j-1}([n]+[\beta])}; e_k\right] z^j. \end{aligned} \tag{2.2}$$

Using relation (1.1) and taking $f = e_k$, $g = e_1$ and $x_j = \frac{q^{j-1}[\alpha]+[j]}{q^{j-1}([n]+[\beta])}$, we obtain

$$\begin{aligned} & \left[\frac{[\alpha]}{[n]+[\beta]}, \ldots, \frac{q^{j-1}[\alpha]+[j]}{q^{j-1}([n]+[\beta])}; e_{k+1}\right] \\ & \quad = \frac{q^{j-1}[\alpha]+[j]}{q^{j-1}([n]+[\beta])} \left[\frac{[\alpha]}{[n]+[\beta]}, \ldots, \frac{q^{j-1}[\alpha]+[j]}{q^{j-1}([n]+[\beta])}; e_k\right] \\ & \qquad + \left[\frac{[\alpha]}{[n]+[\beta]}, \ldots, \frac{q^{j-2}[\alpha]+[j-1]}{q^{j-2}([n]+[\beta])}; e_k\right], \end{aligned} \tag{2.3}$$

using this in $T_{n,k+1}^{\alpha,\beta}(z)$ we reach

$$T_{n,k+1}^{\alpha,\beta}(z) = \frac{qz(1+\frac{z}{q})}{[n]+[\beta]} D_q T_{n,k}^{\alpha,\beta}\left(\frac{z}{q}\right) + \frac{[n]z+[\alpha]}{[n]+[\beta]} T_{n,k}^{\alpha,\beta}(z). \qquad \square$$

Lemma 2 *Let* α *and* β *satisfy* $0 \leq \alpha \leq \beta$. *Denoting* $e_j(z) = z^j$ *and* $W_{n,q}^{0,0}(e_j)$ *by* $W_{n,q}(e_j)$ *given in* (1.4), *for all* $n, k \in \mathbb{N}^0$, *we have the following recursive relation for the images of monomials* e_k *under* $W_{n,q}^{\alpha,\beta}$ *in terms of* $W_{n,q}(e_j)$, $j = 0, 1, \ldots, k$:

$$T_{n,k}^{\alpha,\beta}(z) = \sum_{j=0}^{k} \binom{k}{j} \frac{[n]^j[\alpha]^{k-j}}{([n]+[\beta])^k} W_{n,q}(e_j, z). \tag{2.4}$$

Proof We can use mathematical induction with respect to k. For $k = 0$, equality (2.4) holds. Let it be true for $k = m$, namely

$$T_{n,m}^{\alpha,\beta}(z) = \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^j[\alpha]^{m-j}}{([n]+[\beta])^m} W_{n,q}(e_j, z).$$

Using (2.1), we have

$$\begin{aligned} T_{n,m+1}^{\alpha,\beta}(z) &= \frac{qz(1+\frac{z}{q})}{[n]+[\beta]} \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^j[\alpha]^{m-j}}{([n]+[\beta])^m} D_q W_{n,q}\left(e_j, \frac{z}{q}\right) \\ &\quad + \frac{[n]z+[\alpha]}{[n]+[\beta]} \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^j[\alpha]^{m-j}}{([n]+[\beta])^m} W_{n,q}(e_j, z) \\ &= \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^{j+1}[\alpha]^{m-j}}{([n]+[\beta])^{m+1}} \\ &\quad \times \left[\frac{qz(1+\frac{z}{q})}{[n]} D_q W_{n,q}\left(e_j, \frac{z}{q}\right) + \frac{[n]z+[\alpha]}{[n]} W_{n,q}(e_j, z) \right]. \end{aligned}$$

Taking into account the recurrence relation for the complex q-Baskakov operator in Lemma 2 in [13], we get

$$W_{n,q}(e_{j+1}, z) = \frac{qz(1+\frac{z}{q})}{[n]} D_q W_{n,q}\left(e_j, \frac{z}{q}\right) + zW_{n,q}(e_j, z),$$

which implies

$$\begin{aligned} T_{n,m+1}^{\alpha,\beta}(z) &= \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^{j+1}[\alpha]^{m-j}}{([n]+[\beta])^{m+1}} \left[W_{n,q}(e_{j+1}, z) + \frac{[\alpha]}{[n]} W_{n,q}(e_j, z) \right] \\ &= \sum_{j=1}^{m} \binom{m}{j-1} \frac{[n]^j[\alpha]^{m-j+1}}{([n]+[\beta])^{m+1}} W_{n,q}(e_j, z) \\ &\quad + \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^j[\alpha]^{m-j+1}}{([n]+[\beta])^{m+1}} W_{n,q}(e_j, z) \\ &= \sum_{j=0}^{m+1} \binom{m+1}{j} \frac{[n]^j[\alpha]^{m-j+1}}{([n]+[\beta])^{m+1}} W_{n,q}(e_j, z), \end{aligned}$$

which proves the lemma. □

3 Approximation by a complex q-Baskakov-Stancu operator

In this section, we give quantitative estimates concerning approximation with the following theorem.

Theorem 1 *For* $1 < R < \infty$, *let*

$$f : \overline{D}_R \cup [R, \infty) \to \mathbb{C}$$

be a function with all its derivatives bounded in $[0,\infty)$ by the same positive constant, analytic in D_R, namely $f(z)=\sum_{k=0}^{\infty}c_k z^k$ for all $z\in D_R$ and suppose that there exist $M>0$ and $A\in(\frac{1}{R},1)$, with the property $|c_k|\le\frac{MA^k}{k!}$ for all $k=0,1,\ldots$ (which implies $|f(z)|\le Me^{A|z|}$ for all $z\in D_R$).

Let $0\le\alpha\le\beta$, $q>1$ and $1\le r<\frac{1}{A}$ be arbitrary but fixed. Then, for all $|z|\le r$ and $n\in\mathbb{N}$, we have

$$\begin{aligned}
&\left|W_{n,q}^{\alpha,\beta}(f)(z)-f(z)\right| \\
&\quad\le\frac{M_{1,r}(f)}{[n]+[\beta]}+\frac{[\beta]}{[n]+[\beta]}M_{2,r}(f)+\frac{[\alpha]}{[n]+[\beta]}M_{3,r}(f) \\
&\quad=M_{r,\alpha,\beta}(f)
\end{aligned}$$

with

$$M_{1,r}(f)=6\sum_{k=2}^{\infty}|c_k|(k+1)!(k-1)r^k<\infty,$$

$$M_{2,r}(f)=\sum_{k=1}^{\infty}|c_k|kr^k<\infty,\qquad M_{3,r}(f)=\sum_{k=1}^{\infty}|c_k|kr^{k-1}<\infty.$$

Proof Using (2.1), one can obtain

$$\begin{aligned}
T_{n,k}^{\alpha,\beta}(z)-z^k&=\frac{qz(1+\frac{z}{q})}{[n]+[\beta]}D_q\left(T_{n,k-1}^{\alpha,\beta}\left(\frac{z}{q}\right)\right)+\frac{[n]z+[\alpha]}{[n]+[\beta]}\left(T_{n,k-1}^{\alpha,\beta}(z)-z^{k-1}\right) \\
&\quad+\frac{[n]z+[\alpha]}{[n]+[\beta]}z^{k-1}-z^k \\
&=\frac{z(1+\frac{z}{q})}{[n]+[\beta]}qD_q\left(T_{n,k-1}^{\alpha,\beta}\left(\frac{z}{q}\right)\right)+\frac{[n]z+[\alpha]}{[n]+[\beta]}\left(T_{n,k-1}^{\alpha,\beta}(z)-z^{k-1}\right) \\
&\quad+\left(\frac{[n]}{[n]+[\beta]}-1\right)z^k+\frac{[\alpha]}{[n]+[\beta]}z^{k-1}.
\end{aligned}$$

Moreover, we have

$$qD_q\left(T_{n,k-1}^{\alpha,\beta}\left(\frac{z}{q}\right)\right)=\left|D_q\left(T_{n,k-1}^{\alpha,\beta}(w)\right)\right|_{w=\frac{z}{q}}. \tag{3.1}$$

Now from (3.1) and the Bernstein inequality (see [1]), we have

$$qD_q\left(T_{n,k-1}^{\alpha,\beta}\left(\frac{z}{q}\right)\right)=\left|D_q\left(T_{n,k-1}^{\alpha,\beta}(z)\right)\right|\le\left|T_{n,k-1}^{\prime\alpha,\beta}(z)\right|\le\frac{k-1}{r}\left\|T_{n,k-1}^{\alpha,\beta}\right\|_r,$$

where $\|\cdot\|_r$ is the standard maximum norm over $D_r=\{z\in\mathbb{C}:|z|\le r\}$. Passing to modulus for all $|z|\le r$ and $n\in\mathbb{N}$, we have that

$$\begin{aligned}
\left|T_{n,k}^{\alpha,\beta}(z)-z^k\right|&\le\frac{r(1+r)}{[n]+[\beta]}\left(\frac{k-1}{r}\right)\left\|T_{n,k-1}^{\alpha,\beta}\right\|_r+\frac{[n]r+[\alpha]}{[n]+[\beta]}\left|T_{n,k-1}^{\alpha,\beta}(z)-z^{k-1}\right| \\
&\quad+\left(\frac{[n]}{[n]+[\beta]}-1\right)r^k+\frac{[\alpha]}{[n]+[\beta]}r^{k-1}.
\end{aligned} \tag{3.2}$$

In order to get an estimate for $\|T_{n,k-1}^{\alpha,\beta}\|_r$ in (3.2), we use the following fact:

$$T_{n,k}^{\alpha,\beta}(z) = \sum_{j=0}^{k} \frac{[n][n+1]\cdots[n+j-1]}{([n]+[\beta])^j} q^{\frac{-j(j-1)}{2}} \left[\frac{[\alpha]}{[n]+[\beta]}, \ldots, \frac{q^{j-1}[\alpha]+[j]}{q^{j-1}([n]+[\beta])}; e_k\right] z^j$$

for $k \in \mathbb{N}$. Taking into account Lemma 1 in [13] for $q > 1$, $|z| \le r$, $r \ge 1$ and (1.3), we have

$$\begin{aligned} \left\| T_{n,k}^{\alpha,\beta}(z) \right\|_r &\le r^j \sum_{j=0}^{k} \frac{[n][n+1]\cdots[n+j-1]}{[n]^j} q^{\frac{-j(j-1)}{2}} \\ &\quad \times \left[\frac{[\alpha]}{[n]+[\beta]}, \ldots, \frac{q^{j-1}[\alpha]+[j]}{q^{j-1}([n]+[\beta])}; e_k\right] \\ &\le \sum_{j=0}^{k} j! \frac{kk-1\cdots k-j+1}{j!} r^{k-j} \cdot r^j \\ &= r^k \sum_{j=0}^{k} kk-1\cdots k-j+1 \le r^k (k+1)!. \end{aligned} \tag{3.3}$$

Now, considering (3.3) in (3.2), for all $|z| \le r$, $r \ge 1$, with $q > 1$ and $0 \le \alpha \le \beta$,

$$\begin{aligned} &\left| T_{n,k}^{\alpha,\beta}(z) - z^k \right| \\ &\quad \le \frac{r(1+r)}{[n]+[\beta]} r^{k-2}(k+1)! + \frac{[n]r+[\alpha]}{[n]+[\beta]} \left| T_{n,k-1}^{\alpha,\beta}(z) - z^{k-1} \right| \\ &\qquad + \left(\frac{[n]}{[n]+[\beta]} - 1 \right) r^k + \frac{[\alpha]}{[n]+[\beta]} r^{k-1} \\ &\quad \le \frac{[n]r+[\alpha]}{[n]+[\beta]} \left| T_{n,k-1}^{\alpha,\beta}(z) - z^{k-1} \right| + \frac{r(1+r)}{[n]+[\beta]} r^{k-2}(k+1)! \\ &\qquad + \frac{[\beta]}{[n]+[\beta]} r^k + \frac{[\alpha]}{[n]+[\beta]} r^{k-1} \\ &\quad \le r \left| T_{n,k-1}^{\alpha,\beta}(z) - z^{k-1} \right| + \frac{2r^k}{[n]+[\beta]}(k+1)! \\ &\qquad + \frac{[\beta]}{[n]+[\beta]} r^k + \frac{[\alpha]}{[n]+[\beta]} r^{k-1}. \end{aligned} \tag{3.4}$$

Using the above inequalities beginning from $k = 2, 3, \ldots$ and using the mathematical induction with respect to k, we arrive at

$$\begin{aligned} &\left| T_{n,k}^{\alpha,\beta}(z) - z^k \right| \\ &\quad \le \frac{2r^k}{[n]+[\beta]} \sum_{j=2}^{k} (j+1)! + \frac{[\beta]}{[n]+[\beta]} kr^k + \frac{[\alpha]}{[n]+[\beta]} kr^{k-1} \\ &\quad \le \frac{6r^k}{[n]+[\beta]}(k+1)!(k-1) + \frac{[\beta]}{[n]+[\beta]} kr^k + \frac{[\alpha]}{[n]+[\beta]} kr^{k-1}. \end{aligned} \tag{3.5}$$

Also we obtain the following: for $k = 1$ it is not difficult to see that

$$\left|T_{n,1}^{\alpha,\beta}(z) - z\right| = \left|\frac{[\alpha] - [\beta]z}{[n] + [\beta]}\right| \le \frac{[\alpha] + [\beta]r}{[n] + [\beta]}.$$

Now, taking into account the proof of Theorem 1 in [13], we can write, for $q > 1$, $|z| \le r$, $r \ge 1$, that

$$W_{n,q}^{\alpha,\beta}(f)(z) = \sum_{k=0}^{\infty} c_k T_{n,k}^{\alpha,\beta}(z),$$

which implies

$$\begin{aligned}
&\left|W_{n,q}^{\alpha,\beta}(f)(z) - f(z)\right| \\
&\quad \le \sum_{k=1}^{\infty} |c_k| \left|T_{n,k}^{\alpha,\beta}(z) - z^k\right| \\
&\quad \le \frac{6}{[n] + [\beta]} \sum_{k=1}^{\infty} |c_k|(k+1)!(k-1)r^k + \frac{[\beta]}{[n] + [\beta]} \sum_{k=1}^{\infty} |c_k| k r^k \\
&\qquad + \frac{[\alpha]}{[n] + [\beta]} \sum_{k=1}^{\infty} |c_k| k r^{k-1} \\
&\quad = \frac{M_{1,r}(f)}{[n] + [\beta]} + \frac{[\beta]}{[n] + [\beta]} M_{2,r}(f) + \frac{[\alpha]}{[n] + [\beta]} M_{3,r}(f).
\end{aligned}$$

Here from the analyticity of f we have $M_{2,r}(f) < \infty$ and $M_{3,r}(f) < \infty$. Also from the hypotheses of the theorem, one can get

$$M_{1,r}(f) = 6 \sum_{k=1}^{\infty} |c_k|(k+1)!(k-1)r^k \le 6M \sum_{k=1}^{\infty} (k+1)(k-1)(rA)^k$$

for all $|z| \le r$, $1 \le r \le \frac{1}{A}$, $n \in \mathbb{N}$. □

Theorem 2 *Let* $0 \le \alpha \le \beta$, $1 \le r \le \frac{1}{A}$ *and* $q > 1$. *Under the hypotheses of Theorem* 1, *for all* $|z| \le r$ *and* $n \in \mathbb{N}$, *the following Voronovskaja-type result*

$$\begin{aligned}
&\left|W_{n,q}^{\alpha,\beta}(f)(z) - f(z) - \frac{[\alpha] - [\beta]z}{[n] + [\beta]} f'(z) - \frac{z}{2[n]}\left(1 + \frac{z}{q}\right) f''(z)\right| \\
&\quad \le \frac{K_{1,r}(f)}{[n]^2} + \frac{\sum_{j=2}^{6} K_{j,r}(f)}{([n] + [\beta])^2}
\end{aligned}$$

holds with

$$K_{1,r}(f) = 16 \sum_{k=3}^{\infty} |c_k|(k-1)(k-2)^2 k! r^k < \infty,$$

$$K_{2,r}(f) = [\alpha]^2 \sum_{k=2}^{\infty} |c_k| \frac{(k-1)k!}{2} r^{k-2} < \infty,$$

$$K_{3,r}(f) = 6[\alpha]\sum_{k=2}^{\infty}|c_k|k^2k!r^{k-1} < \infty,$$

$$K_{4,r}(f) = \left(\frac{[\beta]^2}{2} + 6[\beta]\right)\sum_{k=0}^{\infty}|c_k|k^2(k+1)!r^k < \infty,$$

$$K_{5,r}(f) = [\alpha][\beta]\sum_{k=0}^{\infty}|c_k|k(k-1)r^{k-1} < \infty,$$

$$K_{6,r}(f) = [\beta]^2\sum_{k=0}^{\infty}|c_k|k(k-1)r^k < \infty.$$

Proof For all $z \in D_R$, let us consider

$$\begin{aligned} &W_{n,q}^{\alpha,\beta}(f)(z) - f(z) - \frac{[\alpha]-[\beta]z}{[n]+[\beta]}f'(z) - \frac{z}{2[n]}\left(1+\frac{z}{q}\right)f''(z) \\ &\quad = W_{n,q}(f)(z) - f(z) - \frac{z}{2[n]}\left(1+\frac{z}{q}\right)f''(z) \\ &\qquad + W_{n,q}^{\alpha,\beta}(f)(z) - W_{n,q}(f)(z) - \frac{[\alpha]-[\beta]z}{[n]+[\beta]}f'(z). \end{aligned}$$

Using the fact that $f(z) = \sum_{k=0}^{\infty} c_k z^k$, we get

$$\begin{aligned} &W_{n,q}^{\alpha,\beta}(f)(z) - f(z) - \frac{[\alpha]-[\beta]z}{[n]+[\beta]}f'(z) - \frac{z}{2[n]}\left(1+\frac{z}{q}\right)f''(z) \\ &\quad = \sum_{k=2}^{\infty} c_k\left(W_{n,q}(e_k;z) - z^k - \frac{z}{2[n]}\left(1+\frac{z}{q}\right)k(k-1)z^{k-2}\right) \\ &\qquad + \sum_{k=2}^{\infty} c_k\left(T_{n,k}^{\alpha,\beta}(z) - W_{n,q}(e_k;z) - \frac{[\alpha]-[\beta]z}{[n]+[\beta]}kz^{k-1}\right). \end{aligned}$$

From Theorem 2 in [13], we have

$$\begin{aligned} &\left|W_{n,q}(f)(z) - f(z) - \frac{z}{2[n]}\left(1+\frac{z}{q}\right)f''(z)\right| \\ &\quad \le \frac{16}{[n]^2}\sum_{k=3}^{\infty}|c_k|(k-1)(k-2)^2k!r^k. \end{aligned}$$

Furthermore, in order to estimate the second sum, using Lemma 2, we obtain

$$\begin{aligned} &T_{n,k}^{\alpha,\beta}(z) - W_{n,q}(e_k;z) - \frac{[\alpha]-[\beta]z}{[n]+[\beta]}kz^{k-1} \\ &\quad = \sum_{j=0}^{k}\binom{k}{j}\frac{[n]^j[\alpha]^{k-j}}{([n]+[\beta])^k}W_{n,q}(e_j;z) - W_{n,q}(e_k;z) - \frac{[\alpha]-[\beta]z}{[n]+[\beta]}kz^{k-1} \\ &\quad = \sum_{j=0}^{k-1}\binom{k}{j}\frac{[n]^j[\alpha]^{k-j}}{([n]+[\beta])^k}W_{n,q}(e_j;z) \\ &\qquad + \left(\frac{[n]^k}{([n]+[\beta])^k} - 1\right)W_{n,q}(e_k;z) - \frac{[\alpha]-[\beta]z}{[n]+[\beta]}kz^{k-1}. \end{aligned}$$

Also it is clear that

$$1-\frac{[n]^k}{([n]+[\beta])^k}=\sum_{j=1}^{k-1}\binom{k}{j}\frac{[n]^j[\beta]^{k-j}}{([n]+[\beta])^k}\leq\sum_{j=1}^{k-1}\left(1-\frac{[n]}{[n]+[\beta]}\right)=\frac{k[\beta]}{[n]+[\beta]},\tag{3.6}$$

which implies

$$\begin{aligned}
&T_{n,k}^{\alpha,\beta}(z)-W_{n,q}(e_k;z)-\frac{[\alpha]-[\beta]z}{[n]+[\beta]}kz^{k-1}\\
&\quad=\sum_{j=0}^{k-2}\binom{k}{j}\frac{[n]^j[\alpha]^{k-j}}{([n]+[\beta])^k}W_{n,q}(e_j;z)+\frac{k[n]^{k-1}[\alpha]}{([n]+[\beta])^k}W_{n,q}(e_{k-1};z)\\
&\qquad-\sum_{j=0}^{k-2}\binom{k}{j}\frac{[n]^j[\beta]^{k-j}}{([n]+[\beta])^k}W_{n,q}(e_k;z)-\frac{[\alpha]-[\beta]z}{[n]+[\beta]}kz^{k-1}\\
&\quad=\sum_{j=0}^{k-2}\binom{k}{j}\frac{[n]^j[\alpha]^{k-j}}{([n]+[\beta])^k}W_{n,q}(e_j;z)+\frac{k[n]^{k-1}[\alpha]}{([n]+[\beta])^k}\left(W_{n,q}(e_{k-1};z)-z^{k-1}\right)\\
&\qquad-\sum_{j=0}^{k-2}\binom{k}{j}\frac{[n]^j[\beta]^{k-j}}{([n]+[\beta])^k}W_{n,q}(e_k;z)-\frac{k[n]^{k-1}[\beta]}{([n]+[\beta])^k}\left(W_{n,q}(e_k;z)-z^k\right)\\
&\qquad+\left(\frac{[n]^{k-1}}{([n]+[\beta])^{k-1}}-1\right)\frac{k[\alpha]}{[n]+[\beta]}z^{k-1}\\
&\qquad+\left(1-\frac{[n]^{k-1}}{([n]+[\beta])^{k-1}}\right)\frac{k[\beta]}{[n]+[\beta]}z^k.
\end{aligned}\tag{3.7}$$

Now from (3.3) we obtain

$$\begin{aligned}
&\left|\sum_{j=0}^{k-2}\binom{k}{j}\frac{[n]^j[\alpha]^{k-j}}{([n]+[\beta])^k}W_{n,q}(e_j;z)\right|\\
&\quad\leq\sum_{j=0}^{k-2}\binom{k}{j}\frac{[n]^j[\alpha]^{k-j}}{([n]+[\beta])^k}\left|W_{n,q}(e_j;z)\right|\\
&\quad=\sum_{j=0}^{k-2}\frac{k(k-1)}{(k-j)(k-j-1)}\binom{k-2}{j}\frac{[n]^j[\alpha]^{k-j}}{([n]+[\beta])^k}\left|W_{n,q}(e_j;z)\right|\\
&\quad\leq\frac{k(k-1)}{2}\frac{[\alpha]^2}{([n]+[\beta])^2}r^{k-2}(k-1)!\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{[n]^j[\alpha]^{k-2-j}}{([n]+[\beta])^{k-2}}\\
&\quad\leq\frac{k(k-1)}{2}\frac{[\alpha]^2}{([n]+[\beta])^2}r^{k-2}(k-1)!.
\end{aligned}\tag{3.8}$$

Also, we need to prove the following inequality:

$$\begin{aligned}
\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{[n]^j[\alpha]^{k-2-j}}{([n]+[\beta])^{k-2}}&=\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{[n]^j}{([n]+[\beta])^j}\frac{[\alpha]^{k-2-j}}{([n]+[\beta])^{k-2-j}}\\
&=\left(\frac{[n]+[\alpha]}{[n]+[\beta]}\right)^{k-2}\leq 1.
\end{aligned}\tag{3.9}$$

Moreover, taking $\alpha = \beta = 0$ in Theorem 1, we have

$$\left|W_{n,q}(e_k;z) - z^k\right| \le \frac{6}{[n]} r^k (k+1)!(k-1). \tag{3.10}$$

Writing (3.8), (3.6), (3.9) and (3.10) in (3.7), we have

$$\begin{aligned}
&\left|T_{n,k}^{\alpha,\beta}(z) - W_{n,q}(e_k;z) - \frac{[\alpha]-[\beta]z}{[n]+[\beta]} k z^{k-1}\right| \\
&\quad \le \left|\sum_{j=0}^{k-2}\binom{k}{j}\frac{[n]^j[\alpha]^{k-j}}{([n]+[\beta])^k} W_{n,q}(e_j;z)\right| + \frac{k[n]^{k-1}[\alpha]}{([n]+[\beta])^k}\left|W_{n,q}(e_{k-1};z) - z^{k-1}\right| \\
&\qquad + \left|\sum_{j=0}^{k-2}\binom{k}{j}\frac{[n]^j[\beta]^{k-j}}{([n]+[\beta])^k} W_{n,q}(e_k;z)\right| + \frac{k[n]^{k-1}[\beta]}{([n]+[\beta])^k}\left|W_{n,q}(e_k;z) - z^k\right| \\
&\qquad + \left|\frac{[n]^{k-1}}{([n]+[\beta])^{k-1}} - 1\right| \frac{k[\alpha]}{[n]+[\beta]}|z|^{k-1} + \left|1 - \frac{[n]^{k-1}}{([n]+[\beta])^{k-1}}\right| \frac{k[\beta]}{[n]+[\beta]}|z|^k \\
&\quad \le \frac{(k-1)k!}{2}\frac{[\alpha]^2}{([n]+[\beta])^2} r^{k-2} + \frac{k[n]^{k-1}[\alpha]}{([n]+[\beta])^k}\frac{6}{[n]} r^{k-1} k!(k-2) \\
&\qquad + r^k (k+1)! \sum_{j=0}^{k-2}\binom{k}{j}\frac{[n]^j[\beta]^{k-j}}{([n]+[\beta])^k} \\
&\qquad + \frac{k[n]^{k-1}[\beta]}{([n]+[\beta])^k}\frac{6}{[n]} r^k (k+1)!(k-1) + \frac{k(k-1)[\alpha][\beta]}{([n]+[\beta])^2} r^{k-1} + \frac{k(k-1)[\beta]^2}{([n]+[\beta])^2} r^k \\
&\quad \le \frac{(k-1)k!}{2}\frac{[\alpha]^2}{([n]+[\beta])^2} r^{k-2} + 6\frac{k^2[\alpha]}{([n]+[\beta])^2} r^{k-1} k! + \frac{k^2[\beta]^2(k+1)!}{2([n]+[\beta])^2} r^k \\
&\qquad + 6\frac{k^2(k+1)![\beta]}{([n]+[\beta])^2} r^k + \frac{k(k-1)[\alpha][\beta]}{([n]+[\beta])^2} r^{k-1} + \frac{k(k-1)[\beta]^2}{([n]+[\beta])^2} r^k \\
&\quad \le \frac{(k-1)k!}{2}\frac{[\alpha]^2}{([n]+[\beta])^2} r^{k-2} + 6\frac{k^2[\alpha]}{([n]+[\beta])^2} r^{k-1} k! \\
&\qquad + \left(\frac{[\beta]^2}{2} + 6[\beta]\right)\frac{k^2(k+1)!}{([n]+[\beta])^2} r^k \\
&\qquad + \frac{k(k-1)[\alpha][\beta]}{([n]+[\beta])^2} r^{k-1} + \frac{k(k-1)[\beta]^2}{([n]+[\beta])^2} r^k.
\end{aligned}$$

Thus the proof is completed. □

Now, let us give a lower estimate for the exact degree in approximation by $W_{n,q}^{\alpha,\beta}$.

Theorem 3 *Suppose that $q > 1$ and suppose that the hypotheses on f and on the constants R, M, A in the statement of Theorem 1 hold, and let $1 \le r < R$, $0 \le \alpha \le \beta$. If f is not a polynomial of degree ≤ 0, then the lower estimate*

$$\left\|W_{n,q}^{\alpha,\beta}(f) - f\right\|_r \ge \frac{C_r^{\alpha,\beta}(f)}{[n]}$$

holds for all n, where the constant $C_r^{\alpha,\beta}(f)$ depends on f, α, β, q and r.

Proof For all $|z| \leq r$ and $n \in \mathbb{N}$, we get

$$\begin{aligned} & W_{n,q}^{\alpha,\beta}(f)(z) - f(z) \\ & = \frac{1}{[n]} \left\{ \frac{[n]}{[n]+[\beta]} \big([\alpha] - [\beta]z\big) f'(z) + \frac{z}{2}\left(1 + \frac{z}{q}\right) f''(z) \right. \\ & \quad \left. + \frac{1}{[n]}[n]^2 \left(W_{n,q}^{\alpha,\beta}(f)(z) - f(z) - \frac{[\alpha]-[\beta]z}{[n]+[\beta]} f'(z) - \frac{z}{2[n]}\left(1+\frac{z}{q}\right) f''(z) \right) \right\} \\ & = \frac{1}{[n]} \left\{ \big([\alpha] - [\beta]z\big) f'(z) + \frac{z}{2}\left(1 + \frac{z}{q}\right) f''(z) \right. \\ & \quad + \frac{1}{[n]}[n]^2 \left(W_{n,q}^{\alpha,\beta}(f)(z) - f(z) - \frac{[\alpha]-[\beta]z}{[n]+[\beta]} f'(z) \right) \\ & \quad \left. + \frac{1}{[n]}[n]^2 \left(-\frac{z}{2[n]}\left(1+\frac{z}{q}\right) f''(z) - \frac{[\beta]([\alpha]-[\beta]z)}{[n]([n]+[\beta])} f'(z) \right) \right\}. \end{aligned}$$

We set $E_{k,n}(z)$ by

$$\begin{aligned} E_{k,n}(z) := {} & W_{n,q}^{\alpha,\beta}(f)(z) - f(z) - \frac{[\alpha]-[\beta]z}{[n]+[\beta]} f'(z) \\ & - \frac{z}{2[n]}\left(1+\frac{z}{q}\right) f''(z) - \frac{[\beta]([\alpha]-[\beta]z)}{[n]([n]+[\beta])} f'(z). \end{aligned} \tag{3.11}$$

Passing to the norm and using the inequality

$$\|F+G\|_r \geq \big| \|F\|_r - \|G\|_r \big| \geq \|F\|_r - \|G\|_r,$$

we get

$$\big\| W_{n,q}^{\alpha,\beta}(f) - f \big\|_r \geq \frac{1}{[n]} \left\| \big([\alpha] - [\beta]e_1\big) f' + \frac{e_1}{2}\left(1 + \frac{e_1}{q}\right) f'' \right\|_r - \frac{1}{[n]}[n]^2 \|E_{k,n}\|_r.$$

Since f is not a polynomial of degree ≤ 0 in D_R, we have

$$\left\| \big([\alpha] - [\beta]e_1\big) f' + \frac{e_1}{2}\left(1 + \frac{e_1}{q}\right) f'' \right\|_r > 0.$$

It can also be seen in [1, pp.75-76]. Now, from Theorem 2 it follows that

$$\begin{aligned} [n]^2 \|E_{k,n}\|_r & \leq [n]^2 \left\| W_{n,q}^{\alpha,\beta}(f) - f - \left(\frac{[\alpha]-[\beta]e_1}{[n]+[\beta]} \right) f' - \frac{e_1}{2[n]}\left(1+\frac{e_1}{q}\right) f'' \right\|_r \\ & \quad + \big\| [\beta]\big([\alpha] - [\beta]e_1\big) f' \big\|_r \\ & \leq \sum_{j=1}^{6} M_{j,r}(f) + [\beta]\big([\alpha] + [\beta]r\big) \big\| f' \big\|_r. \end{aligned}$$

Since $\frac{1}{[n]} \to 0$ as $n \to \infty$, for $q > 1$, there exists an n_0 depending on f, r, α, β and q such that for all $n \geq n_0$,

$$\begin{aligned} &\frac{1}{[n]} \left\| \left([\alpha] - [\beta]e_1\right)f' + \frac{e_1}{2}\left(1 + \frac{e_1}{q}\right)f'' \right\|_r - \frac{1}{[n]}[n]^2 \|E_{k,n}\|_r \\ &\quad \geq \frac{1}{2} \left\| \left([\alpha] - [\beta]e_1\right)f' + \frac{e_1}{2}\left(1 + \frac{e_1}{q}\right)f'' \right\|_r, \end{aligned}$$

which implies

$$\left\| W_{n,q}^{\alpha,\beta}(f) - f \right\|_r \geq \frac{1}{2[n]} \left\| \left([\alpha] - [\beta]e_1\right)f' + \frac{e_1}{2}\left(1 + \frac{e_1}{q}\right)f'' \right\|_r$$

for all $n \geq n_0$. Now, for $n \in \{1, \ldots, n_0 - 1\}$, we have

$$\left\| W_{n,q}^{\alpha,\beta}(f) - f \right\|_r \geq \frac{A_r(f)}{[n]}$$

with

$$A_r(f) = [n] \left\| W_{n,q}^{\alpha,\beta}(f) - f \right\|_r > 0,$$

which finally implies

$$\left\| W_{n,q}^{\alpha,\beta}(f) - f \right\|_r \geq \frac{C_r^{\alpha,\beta}(f)}{[n]}$$

for all $n \geq n_0$ with

$$C_r^{\alpha,\beta}(f) = \min \left\{ A_{r,1}(f), \ldots, A_{r,n_0-1}(f), \frac{1}{2} \left\| \left([\alpha] - [\beta]e_1\right)f' + \frac{e_1}{2}\left(1 + \frac{e_1}{q}\right)f'' \right\|_r \right\}.$$

This proves the theorem. □

Combining now Theorem 3 with Theorem 1, we immediately get the following equivalence result.

Remark 1 Suppose that $q > 1$, $0 \leq \alpha \leq \beta$ and that the hypotheses on f and on the constants R, M, A in the statement of Theorem 1 hold, and let $1 \leq r < \frac{1}{A}$ be fixed. If f is not a polynomial of degree ≤ 0, then we have the following equivalence:

$$\left\| W_{n,q}^{\alpha,\beta}(f) - f \right\|_r \sim \frac{1}{[n]}$$

for all n, where the constants in the equivalence depend on f, α, β, q and r.

Concerning the approximation by the derivatives of complex q-Baskakov-Stancu operators, we can state the following theorem.

Theorem 4 *Suppose that $q > 1$ and that the hypotheses on f and on the constants R, M, A in the statement of Theorem 1 hold, and let $0 \leq \alpha \leq \beta$, $1 \leq r < r_1 < \frac{1}{A}$ and $p \in \mathbb{N}$ be fixed. If f is not a polynomial of degree $\leq p-1$, then we have the following equivalence*:

$$\left\| \left[W_{n,q}^{\alpha,\beta}(f) \right]^{(p)} - f^{(p)} \right\|_r \sim \frac{1}{[n]}$$

for all n, where the constants in the equivalence depend on f (that is, on M, A), $r, r_1 q$ and p.

Proof Denote by Γ the circle of radius r_1 with $1 \leq r < r_1 < \frac{1}{A}$ centered at 0. Since $|z| \leq r$ and $\gamma \in \Gamma$, we have $|\gamma - z| \geq r_1 - r$ and from Cauchy's formulas and Theorem 1 we obtain, for all $|z| \leq r$ and $n \in \mathbb{N}$, that

$$\begin{aligned} \left| \left[W_{n,q}^{\alpha,\beta}(f,z) \right]^{(p)} - f^{(p)}(z) \right| &\leq \frac{p!}{2\pi} \left| \int_\Gamma \frac{W_{n,q}^{\alpha,\beta} f(\gamma) - f(\gamma)}{(\gamma - z)^{p+1}} \, d\gamma \right| \\ &\leq \frac{M_{r_1,\alpha,\beta}(f)}{[n]} \frac{p!}{2\pi} \frac{2\pi r_1}{(r_1 - r)^{p+1}} \\ &= \frac{M_{r_1,\alpha,\beta}(f)}{[n]} \frac{p! r_1}{(r_1 - r)^{p+1}}, \end{aligned}$$

which proves one of the inequalities in the equivalence.

Now we need to prove the lower estimate. From Cauchy's formula we get

$$\left[W_{n,q}^{\alpha,\beta}(f,z) \right]^{(p)} - f^{(p)}(z) = \frac{p!}{2\pi i} \int_\Gamma \frac{W_{n,q}^{\alpha,\beta} f(\gamma) - f(\gamma)}{(\gamma - z)^{p+1}} \, d\gamma.$$

Furthermore, using (3.11) one can have

$$\begin{aligned} & W_{n,q}^{\alpha,\beta} f(\gamma) - f(\gamma) \\ &\quad = \frac{1}{[n]} \left\{ \left([\alpha] - [\beta]\gamma \right) f'(\gamma) + \frac{\gamma}{2} \left(1 + \frac{\gamma}{q} \right) f''(\gamma) + [n]^2 E_{k,n}(\gamma) \right\} \end{aligned}$$

for all $\gamma \in \Gamma$ and $n \in \mathbb{N}$. Applications of Cauchy's formula imply

$$\begin{aligned} & \left[W_{n,q}^{\alpha,\beta}(f,z) \right]^{(p)} - f^{(p)}(z) \\ &\quad = \left\{ \frac{1}{[n]} \frac{p!}{2\pi i} \int_\Gamma \frac{([\alpha] - [\beta]\gamma) f'(\gamma) + \frac{\gamma}{2}(1 + \frac{\gamma}{q}) f''(\gamma)}{(\gamma - z)^{p+1}} \, d\gamma \right. \\ &\qquad \left. + \frac{1}{[n]} \frac{p!}{2\pi i} \int_\Gamma \frac{[n]^2 E_{k,n}(\gamma)}{(\gamma - z)^{p+1}} \, d\gamma \right\} \\ &\quad = \frac{1}{[n]} \left\{ \left[\left([\alpha] - [\beta]\gamma \right) f'(\gamma) + \frac{z}{2} \left(1 + \frac{z}{q} \right) f''(z) \right]^{(p)} + \frac{p!}{2\pi i} \int_\Gamma \frac{[n]^2 E_{k,n}(\gamma)}{(\gamma - z)^{p+1}} \, d\gamma \right\}. \end{aligned}$$

Now passing to the norm $\| \cdot \|_r$ we obtain

$$\begin{aligned} \left\| \left[W_{n,q}^{\alpha,\beta}(f) \right]^{(p)} - f^{(p)} \right\|_r \geq \frac{1}{[n]} & \left\{ \left\| \left[\left([\alpha] - [\beta] e_1 \right) f' + \frac{e_1}{2} \left(1 + \frac{e_1}{q} \right) f'' \right]^{(p)} \right\|_r \right. \\ & \left. - \frac{1}{[n]} \left\| \frac{p!}{2\pi} \int_\Gamma \frac{[n]^2 E_{k,n}(\gamma)}{(\gamma - z)^{p+1}} \, d\gamma \right\|_r \right\}, \end{aligned}$$

and from Theorem 2 we have

$$\left\| \frac{p!}{2\pi} \int_\Gamma \frac{[n]^2 E_{k,n}(\gamma)}{(\gamma - z)^{p+1}} \, d\gamma \right\|_r \leq \frac{p!}{2\pi} \frac{2\pi r_1}{(r_1 - r)^{p+1}} [n]^2 \|E_{k,n}\|_{r_1}$$
$$\leq K_{1,r_1}(f) + [n]^2 \frac{\sum_{j=2}^{6} K_{j,r_1}(f)}{([n] + [\beta])^2} + [\beta]\big([\alpha] + [\beta] r_1\big) \|f'\|_{r_1}.$$

Since f is not a polynomial of degree ≤ 0 in D_R, we have

$$\left\| \left[\big([\alpha] - [\beta] e_1\big) f' + \frac{e_1}{2} \left(1 + \frac{e_1}{q}\right) f'' \right]^{(p)} \right\|_r > 0$$

(see [1, pp.77-78]). The rest of the proof is obtained similarly to that of Theorem 3. □

Remark 2 Note that if we take $\alpha = \beta = 0$, then Theorems 1, 2, 3 and 4 reduce to the results in [13].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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