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On concomitants of upper record statistics and survival analysis for a pseudo-Gompertz distribution

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ABSTRACT

This paper presents upper record statistics and their concomitants for a bivariate pseudo-Gompertz distribution about paired lifetime variables. Survival and hazard functions are derived for the distribution. The survival and hazard functions are displayed for some selected values of the parameters of concern. Interpretations are given for the potential reliability and actuarial applications of the obtained results.

1. Introduction

Records on a random variable X are realized as a sequence of observations that are larger or smaller than all the previously observed ones in the sequence. So, the record values are distinct elements in the successive maxima or minima about a sequence of a random variable. Every change in the maximum during the observation process for X means that a record is observed. In this regard, records are dealt with in the scope of the extremal processes as discussed in [1]. Many theoretical or applied areas of science use the theory and methodology about the record values in the empirical investigations about the timing and magnitude of record type extremities.

Let $\{(X_i, Y_i), i = 1, \ldots, n\}$ be a random sample from a bivariate distribution function $F(x, y)$ of a random vector (X, Y). The values in the random sample are observed as the realizations of the $\{(X_i, Y_i), i = 1, 2, \ldots\}$ sequence of identically distributed bivariate random variables with a common joint distribution function $F(x, y)$. The $\{X_i\}$ sequence from $\{(X_i, Y_i), i = 1, 2, \ldots\}$ is itself a sequence of random variables with a common distribution function F_X which is the marginal distribution of X from the joint distribution function $F(x, y)$. Further, T indicates the time of observations on (X, Y) such that $T_1 = 1$ and $T_r = \inf\left\{k : k > T_{r-1}, X_k > X_{T_{r-1}}\right\}$. Letting $R_r = X_{T_r}$, the sequence $\{R_r\}$ is defined as the sequence of the upper records of $\{X_i\}$, and the corresponding Y-coordinates Y_{T_r}, denoted by $R_{[r]}$, are defined as the sequence of concomitants $\left\{R_{[r]}\right\}$ of the upper records. Lower record values and their concomitants are defined similarly as seen in [1–3].

The statistical features and potential applications of record values were introduced in the early fifties by Candler [2]. Since then, a rich literature has grown on the statistical theory and methodology for the data analysis and inference about the record values as seen in [1,3–6].

This paper undertakes the random vector (X, Y) to represent paired lifetimes of components of a physical system or lifetimes of human beings in a population. The coupling of Y with X can be expressed by some analytical functions in order

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to associate Y with X up to the investigation and modeling interests of the researchers. Several probability distribution models can be considered for the random lifetimes vector (X, Y). Life distribution models are discussed in depth by Marshall and Olkin [7]. Among the known life distribution models, the Gompertz distribution attracts attention as a suitable model to compute age specific lifetime probabilities and to express hazard rates for system components or mortality rates for individuals in flexible and meaningful ways [8,9]. We consider the bivariate Gompertz distribution for the survival modeling attempts about the paired lifetimes and propose a bivariate pseudo-Gompertz distribution along the lines of pseudo-distributions. A relatively recent work by Filus and Filus [10] presents the definition and theoretical background of the pseudo-distributions.

The following sections of the paper begins with the definitions and distributions of the upper record values and their concomitants. Then, a bivariate pseudo-Gompertz distribution is introduced, and the distributional properties of the upper record values and their concomitants are derived for the new distribution. Thereafter, the survival analysis models are presented and some interpretations and implications about these models are provided. And, a conclusion is given in the sequel.

2. Concomitants of upper record statistics

The theory and applications for the upper record values and their concomitants have been enriched by the recent works of [11,12]. Special functions about the record statistics and concomitants are presented in [13] for the applied scientists. Following them, we denote the upper record statistics and their concomitants, as already written in the first section, with $\left(R_r, R_{[r]}\right) = \left(X_{T_r}, Y_{T_r}\right)$, such that $T_r = \inf\left\{k : k > T_{r-1}, X_k > X_{T_{r-1}}\right\}$. So, it is obvious that $T_1 = 1$ and $\left(R_1, R_{[1]}\right) = (X_1, Y_1)$ with probability one.

Recall that $\{(X_i, Y_i), i = 1, 2, \ldots\}$ is a sequence of identically and independently distributed bivariate random variables with a common distribution $F(x, y)$, and $\{X_i\}$ is a sequence of identically and independently distributed random variables with the common marginal distribution function $F(x)$. Let $f(x)$ denote the marginal probability density function of X. As one can see in [3–5], the probability density function of the r-th upper record R_r is defined as

$$f_{R_r}(x) = \frac{1}{\Gamma(r)} f(x)\left[H(x)\right]^{r-1} \tag{1}$$

where $H(x) = -1n\left[1 - F(x)\right]$. The joint probability density function of the r-th and s-th upper records R_r and R_s, $r < s$, is presented by Ahsanullah [3] using the following general expression

$$f_{R_r,R_s}(x_1, x_2) = \frac{h(x_1) f(x_2)}{\Gamma(r)\,\Gamma(s-r)}\left[H(x_1)\right]^{r-1}\left[H(x_2) - H(x_1)\right]^{s-r-1} \tag{2}$$

where $h(x_r) = \frac{d}{dx_r} H(x_r)$ and $-\infty < x_1 < x_2 < \infty$.

The probability density function for the concomitant of the r-th upper record is also given by Ahsanullah [3] as:

$$f_{R_{[r]}}(y) = \int_{-\infty}^{\infty} f(y \mid x) f_{R_r}(x)dx. \tag{3}$$

The joint probability density function of the concomitants of the r-th and s-th upper record values can be computed, as shown by Ahsanullah [3], in the following form

$$f_{R_{[r]},R_{[s]}}(y_1, y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_s} f(y_1 \mid x_1) f(y_2 \mid x_2) f_{R_r,R_s}(x_1, x_2)dx_1dx_2 \tag{4}$$

where $f_{R_r,R_s}(x_1, x_2)$ is expressed in (2).

Under the assumptions of the model, the expressions given above hold for any bivariate distribution $F(x, y)$.

3. Concomitants of record values for bivariate pseudo-Gompertz distribution

A pseudo-distribution is a function that contains linear combinations of the underlying random variables in its parameters and satisfies all properties that are required to be a probability distribution function. A new class of pseudo-distributions is introduced in [10,14] for statistical applications where an actual distribution cannot be used easily. A multivariate pseudo-distribution is presented by Diaz-Garcia et al. [15]. Some new and specific pseudo-distributions and concomitants for them have been proposed and discussed by Shahbaz and Ahmad [16] and Shahbaz et al. [17]. Along similar lines, the ongoing research by the authors of this paper has produced a new bivariate pseudo-Gompertz distribution for a random vector (X, Y) that associates the random variable Y with X through the real valued function $\phi(x) = e^{\mu_1 x} - 1$ with $\mu_1 > 0$ and $x > 0$.

Using the marginal Gompertz density functions for X and Y

$$f(x; \lambda, \mu_1) = \lambda e^{\mu_1 x} \exp\left[-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x} - 1\right)\right], \quad \mu_1 > 0, \lambda > 0, x > 0,$$

$$f(y; \phi(x), \mu_2|x) = \phi(x)\, e^{\mu_2 y} \exp\left[-\frac{\phi(x)}{\mu_2}\left(e^{\mu_2 y} - 1\right)\right], \quad \mu_2 > 0, x > 0, y > 0,$$

respectively, the bivariate pseudo-Gompertz distribution comes out as the compound distribution of X and Y

$$f(x,y)=\lambda\phi(x)e^{\mu_1x}e^{\mu_2y}\exp\left[-\frac{\lambda}{\mu_1}\left(e^{\mu_1x}-1\right)-\frac{\phi(x)}{\mu_2}\left(e^{\mu_2y}-1\right)\right], \tag{5}$$

with $\mu_1>0,\mu_2>0,\lambda>0,\phi(x)>0,y>0,x>0$. This joint density function follows from the definition

$$f(x,y)=f_X(x;\lambda,\mu_1)f_{Y|X=x}(y;\phi(x),\mu_2|x).$$

Placing $\phi(x)=e^{\mu_1x}-1$ in expression (5), it turns out to be

$$f(x,y)=\lambda\left(e^{\mu_1x}-1\right)e^{\mu_1x}e^{\mu_2y}\exp\left[-\left(e^{\mu_1x}-1\right)\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2y}-1)}{\mu_2}\right)\right],\quad \mu_1,\mu_2,\lambda,y,x>0 \tag{6}$$

such that $F(x,y)=\int_0^x\int_0^y f(t,u)\,dtdu$ possesses all the properties of a probability distribution function.

Using Eq. (1), the probability density function for the r-th record value is derived as;

$$f_{R_r}(x)=\frac{\lambda}{\Gamma(r)}e^{\mu_1x}\exp\left[-\frac{\lambda}{\mu_1}\left(e^{\mu_1x}-1\right)\right]\left[\frac{\lambda}{\mu_1}\left(e^{\mu_1x}-1\right)\right]^{r-1}. \tag{7}$$

The conditional density function, derived from expression (6),

$$f(y|x)=\left(e^{\mu_1x}-1\right)e^{\mu_2y}\exp\left[-\frac{(e^{\mu_1x}-1)}{\mu_2}\left(e^{\mu_2y}-1\right)\right],\quad \mu_1,\mu_2,x,y>0,$$

in connection with expression (7), leads to the probability density function of the concomitant of the r-th record value for the bivariate pseudo-Gompertz distribution;

$$\begin{aligned}f_{R_{[r]}}(y)=&\int_0^\infty e^{\mu_2y}\left(e^{\mu_1x}-1\right)\exp\left[-\left(e^{\mu_1x}-1\right)\left(\frac{(e^{\mu_2y}-1)}{\mu_2}\right)\right]\\&\times\frac{\lambda}{\Gamma(r)}e^{\mu_1x}\exp\left[-\frac{\lambda}{\mu_1}\left(e^{\mu_1x}-1\right)\right]\left[\frac{\lambda}{\mu_1}\left(e^{\mu_1x}-1\right)\right]^{r-1}dx.\end{aligned}$$

Clearly, the probability distribution of the concomitant of the r-th order upper record value is

$$f_{R_{[r]}}(y)=\frac{r\lambda^r\mu_1\mu_2^{r+1}e^{\mu_2y}}{(\lambda\mu_2+\mu_1(e^{\mu_2y}-1))^{r+1}},\quad y>0,\lambda,\mu_1,\mu_2>0,r=1,2,\ldots.$$

The distribution function corresponding to this density function is obtained as

$$F_{R_{[r]}}(y)=1-\left(\frac{\lambda}{\mu_1}\right)^r\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2y}-1)}{\mu_2}\right)^{-r},\quad y>0,\lambda,\mu_1,\mu_2>0,r=1,2,\ldots.$$

The expectation for $R_{[r]}$ is

$$E\left(R_{[r]}\right)=\int_0^\infty y\frac{r\lambda^r\mu_1\mu_2^{r+1}e^{\mu_2y}}{(\mu_2\lambda+\mu_1(e^{\mu_2y}-1))^{r+1}}dy$$

which is computed as shown below by using simple integration calculus:

$$\begin{aligned}E\left(R_{[r]}\right)&=r\lambda^r\mu_1\mu_2^{r+1}\frac{1}{\mu_2^2}\frac{1}{\mu_1r}\int_1^\infty\frac{du}{u(\mu_2\lambda-\mu_1+\mu_1u)^r}\\&=r\lambda^r\mu_1\mu_2^{r+1}\frac{1}{\mu_2^2}\frac{1}{\mu_1r(\mu_2\lambda-\mu_1)^r}\int_1^\infty u^{-1}\left(1+\frac{\mu_1}{\mu_2\lambda-\mu_1}u\right)^{-r}du.\end{aligned}$$

We solve the integral expression in this equation by making use of [18, Eq. 3.194-2, p. 315]

$$\int_u^\infty\frac{x^{\mu-1}dx}{(1+\beta x)^v}=\frac{u^{\mu-v}}{\beta^v(v-\mu)}{}_2F_1\left(v,v-\mu;v-\mu+1;\frac{-1}{\beta u}\right),$$

where $u=1$, $\mu=0$, $\frac{\mu_1}{\mu_2\lambda-\mu_1}=\beta$, and $v=r$. So, the expression for the expectation of $R_{[r]}$ becomes

$$\begin{aligned}E\left(R_{[r]}\right)&=r\lambda^r\mu_1\mu_2^{r+1}\frac{1}{\mu_2^2}\frac{1}{\mu_1^{r+1}r^2}{}_2F_1\left(r,r;r+1;\frac{\mu_1-\mu_2\lambda}{\mu_1}\right)\\&=\frac{\mu_2^{r-1}}{r}\left(\frac{\lambda}{\mu_1}\right)^r{}_2F_1\left[r,r;r+1;\frac{\mu_1-\mu_2\lambda}{\mu_1}\right],\quad r,y,\lambda,\mu_1,\mu_2>0,\end{aligned}$$

where ${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$ is the Gauss Hypergeometric function given in [18]. The convergence of the Gauss Hypergeometric series ${}_2F_1$ holds for all $|z| < 1$, as stated by [18, p. 1005].

So, the condition for convergence imposes further restrictions on the values of μ_1, μ_2 and λ.

The variance $\mathrm{Var}\left(R_{[r]}\right) = E\left(R_{[r]}\right)^2 - \left(E\left(R_{[r]}\right)\right)^2$ of the concomitant of the r-th order upper record value is obtained with the computation of $E\left(R_{[r]}^2\right)$. Following the computational steps as similar for $E\left(R_{[r]}\right)$ above, we reach:

$$E\left(R_{[r]}^2\right) = \int_0^{\infty} y^2 \frac{r\lambda^r \mu_1 \mu_2^{r+1} e^{\mu_2 y}}{\left(\mu_2\lambda + \mu_1\left(e^{\mu_2 y} - 1\right)\right)^{r+1}} dy$$

that, after the transformation $e^{\mu_2 y} = u$, $\mu_2 e^{\mu_2 y} dy = du$, turns out to be

$$\begin{aligned} E\left(R_{[r]}^2\right) &= r\lambda^r \mu_1 \mu_2^{r+1} \frac{1}{\mu_2^3} \int_1^{\infty} \frac{(1n(u))^2}{(\mu_2\lambda - \mu_1 + \mu_1 u)^{r+1}} du \\ &= \frac{2\mu_2^{r-2}}{r^2} \left(\frac{\lambda}{\mu_1}\right)^r {}_3F_2\left[\{r, r, r\}, \{1+r, 1+r\}, \frac{\mu_1 - \mu_2\lambda}{\mu_1}\right], \quad r, y, \lambda, \mu_1, \mu_2 > 0. \end{aligned}$$

Here, the last component in the equation is the Generalized Hypergeometric function whose general form is

$${}_pF_q\left(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; z\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \cdots \left(\alpha_p\right)_k}{(\beta_1)_k (\beta_2)_k \cdots \left(\beta_q\right)_k} \frac{z^k}{k!}$$

expressed in [18]. The Generalized Hypergeometric series ${}_pF_q$ converges for all $|z| < 1$ given that $p = q + 1$, [18, p. 1005].

4. Survival and hazard functions for bivariate pseudo-Gompertz distribution

The survival function for a random variable Y is defined as $S(y) = 1 - F_Y(y)$ which is actually the probability of the survival of a component or a live after the age of y. Under the bivariate pseudo-Gompertz distribution, the survival function for the concomitant of the r-th order upper record value is then obtained as

$$S_{R_{[r]}}(y) = \left(\frac{\lambda}{\mu_1}\right)^r \left(\frac{\lambda}{\mu_1} + \frac{\left(e^{\mu_2 y} - 1\right)}{\mu_2}\right)^{-r}, \quad r, y, \lambda, \mu_1, \mu_2 > 0. \tag{8}$$

The hazard rate or mortality rate function is defined in general as $h(y) = f(y)[S(y)]^{-1}$ for a random lifetime variable Y. This function is a first order approximation for the conditional probability of death for a life or a component aged y. The hazard or mortality rate function for the concomitant variable $R_{[r]}$ under the bivariate pseudo-Gompertz distribution is, then, obtained as

$$h_{R_{[r]}}(y) = \frac{f_{R_{[r]}}(y)}{S_{R_{[r]}}(y)} = \frac{r\mu_1\mu_2 e^{\mu_2 y}}{\lambda\mu_2 + \mu_1\left(e^{\mu_2 y} - 1\right)}, \quad r, y, \lambda, \mu_1, \mu_2 > 0. \tag{9}$$

Record values are the successive maxima values that are matters of interest in many lifetime related events. So, they are usually used in the investigations that focus on survivals and deaths as the core of analysis. The age specific lifetime and remaining lifetime analysis can be precisely based on the order value r and the magnitudes of the record and concomitant values x and y, respectively. The behaviors of the survival and hazard functions, in case of the bivariate pseudo-Gompertz distribution, have some particular features. The value of the survival function of the concomitant the $R_{[r]}$ decreases as the r and y values increase, as obvious. However, it is notable that the rate of decline in the survival function, for the given r values, is faster for the increasing values of the distribution parameter λ. On the other hand, the value of the hazard function of the concomitant $R_{[r]}$ increases, naturally, as r increases; but it is also notable here that it assumes larger values as y and λ values decrease. The overall level of values of the hazard function gets lower as λ values increase, and the rate of this decline is higher for small values of y and λ. Fig. 1 below depicts these characteristic behaviors.

An effective utilization of the survival and hazard functions can be pursued particularly in physical system reliability analysis and actuarial modeling for insurance. In these areas, the survival of the systems and the lives, at any time point, are functions of the age specific future lifetimes of the system components or the individuals. The termination of the survival statuses of the system components or the lives are the cause of the costs. Physical systems managers or insurers have to foresee these costs in order to do the commensurate capital allocations.

There exist numerous actuarial valuation and pricing models in the literature concerning the life statuses of the lives that participate in the insurance plans. For these models and their details, we refer to [19–21].

As an example, consider a whole life insurance case with a portfolio of policies that each is written on two lives, and their lifetimes, represented by the random vector (X, Y), are distributed according to the bivariate pseudo-Gompertz distribution introduced in this paper. Assume that for every policy, the policy owner is the first live whose lifetime is denoted by X and the paired live is the second person. Further, assume that the policies provide contingent insurances that pay benefits to the beneficiaries, who are either the first live or other individuals designated in the policy, upon the death of the second live. The

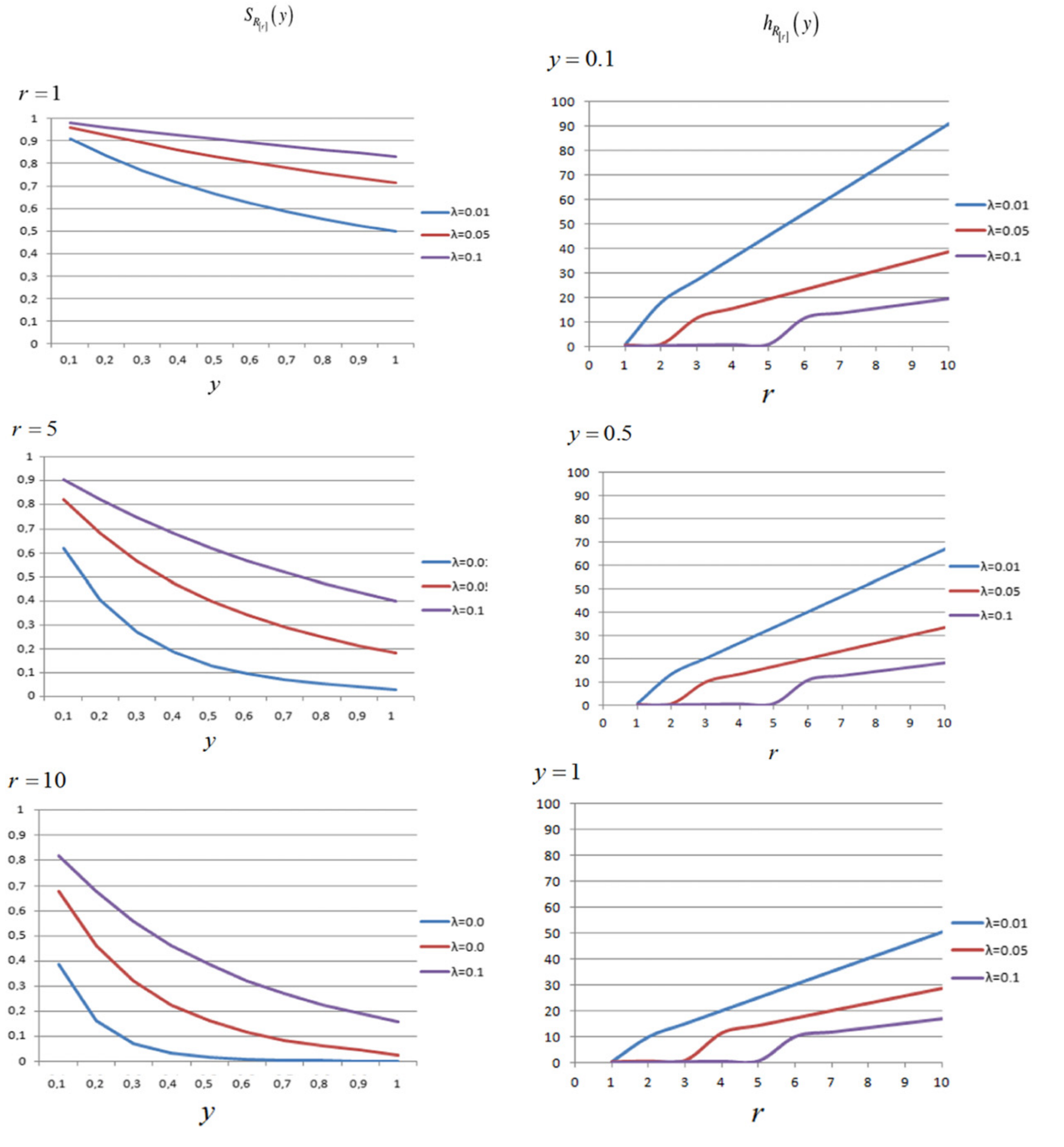

Fig. 1. (a) Survival function for the concomitant $R_{[r]}$ for $r = 1, r = 5, r = 10$. (b) Hazard function for the concomitant $R_{[r]}$ for $y = 0.1, y = 0.5, y = 1$.

reader may see [20] and [21, pp. 135–150] for the concepts and models about the whole life insurance and the contingent insurances. Suppose that an insurer draws a random samples of size n, $\{(X_i, Y_i), i = 1, \ldots, n\}$, from a large population of active whole life insurance policies for the purpose of benefit payment liability assessments. As a matter of insurance liability management strategy, attention may be on the successive maxima of the ages of the active policy owners. Then the focus can be on the record ages of a chosen order r and on its concomitant, $(R_r, R_{[r]}) = (X_{T_r}, Y_{T_r})$, for the aim of securing reserves for the benefit payment liabilities to accrue. The possibility that the insurer fails to meet these kind of liabilities is a risk and it has to be prevented beforehand. In this regard, the whole life insurer is concerned about the survival of the paired lives with ages of (x, y) at the time of the sample where x is the r-th order age of the first live and y is the age of the r-th order concomitant. Due to the contingency of the insurance upon the death of the second live, it is critical to compute the probability that the r-th concomitant live of age y will live for v years more, at least. This probability is expressed as

$$P\left(R_{[r]} > y + v | R_{[r]} > y\right) = S_{R_{[r]}}(y + v) / S_{R_{[r]}}(y) = {}_v p_y,$$

where the survival functions $S(\cdot)$ can be calculated according to Eq. (8), above. The hazard function in (9), with y replaced by $y+v$, expresses the intensity of the probability that a $y+v$ old r-th concomitant live dies immediately. Let $C(y+v)$ be a positive valued function of y and v as the benefit amount to be paid on the contingent death occurrence. At the time of the sample, a life insurer needs to know the required reserves to be available in the next v years. Following [20,21], the actuarial present value of this amount can be calculated by

$$V(y) = \int_0^\infty C(y+v)\, e^{-\delta v} \left({}_v p_y\right) h_{R_{[r]}}(y+v)\, dv,$$

where $h_{R_{[r]}}(y+v)$ is the hazard function for the concomitant live at age $y+v$, δ is the so called force of interest and $e^{-\delta v}$ is the continuously compounding discounting factor with maturity v.

Under the conditions of the given example, the reserve liability calculation for the portfolio can be found by extending this result to the other insurance policies in appropriate ways. For instance, a crude way of such a calculation might be to multiply $V(y)$ by the number of policies. This might be appropriate only if r is so chosen by the insurer that $V(y)$ is believed to represent a typical risk aversive benefit payment liability for any policy in the portfolio.

Note that, the same sort of calculations can be made for the samples independently drawn from two separate but similar populations of the whole life insurance policies in a given time period. So, some comparative studies can be conducted regarding the liabilities that may vary from one population to another. On the other hand, two independent samples from the same population can be drawn in two different periods and they can be compared by the similar calculations. In the latter case, one can measure the variation in the benefit payment liabilities for a population under concern from one period to another.

5. Joint distribution of the concomitants

In some reliability studies, there may be a need for the use of the probability distributions of the concomitants of two record values of order r and s. This section presents the joint distribution of the concomitants of the r-th and s-th order upper record statistics under the pseudo-Gompertz distribution that we have introduced.

A motivation for the derivation of this joint distribution is that it can be used in the reliability analysis of the systems with redundant structures. Redundancy, as defined in [22], is introduced in a system by placing reserve components in the critical positions where the failure of the important single component implies that the system itself fails. If the important component is replaced with at least two components operating in parallel, the resulting redundancy is called an active redundancy such that the system runs as long as at least one of the components operates.

Assume that there exists a large population of electronic devices produced by a company and each device is made of two major components in active redundancy. Suppose that the first component is the product of the company itself and the second one, as a reserve component in active redundancy, is acquired from the market. Suppose also that the company is concerned about the reliable lifetimes of the reserve components along with the record lifetimes of the first components of the electronic devices. A sample survey can be conducted for this kind of an investigation and a random sample of size n of the running electronic devices can be collected with the observations on the ages (X, Y) of the first and the second components, respectively. The joint density function of the concomitant lifetimes $R_{[r]}$ and $R_{[s]}$, $s > r$, that correspond to the r-th and s-th order record lifetimes of the first component, can then be used for the inferential analysis. For instance, the difference $R_{[s]} - R_{[r]}$, that we may call the $s:r$ range of the concomitant lifetimes, can be investigated as an indication of the variability of the lifetimes of the second components. A large $s:r$ range indicates a large variability in the lifetimes of the second components, and this in turn may imply that there exists a large variation in its quality and remaining lifetime reliability. The probability function of the $s:r$ range can be obtained from the joint density function of $R_{[r]}$ and $R_{[s]}$ by the usual methods for the functions of random vectors [23, pp. 127–147]. So, we concentrate below merely on the derivation of the joint density function for $R_{[r]}$ and $R_{[s]}$.

The joint density function in (2) for the record type random variables R_r and R_s with the parent Gompertz distribution is written explicitly below by letting $H(x) = \frac{\lambda}{\mu_1}\left(e^{\mu_1 x} - 1\right)$, $h(x) = \lambda e^{\mu_1 x}$:

$$f_{R_r,R_s}(x_1,x_2) = \frac{\lambda^2 e^{\mu_1(x_1+x_2)}}{\Gamma(r)\,\Gamma(s-r)} \left(\frac{\lambda}{\mu_1}\right)^{s-2} \times \exp\left[-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x_2}-1\right)\right] \left[\left(e^{\mu_1 x_1}-1\right)\right]^{r-1} \left[\left(e^{\mu_1 x_2}-1\right)-\left(e^{\mu_1 x_1}-1\right)\right]^{s-r-1}. \tag{10}$$

Making use of the conditional distribution of Y given X, given in Section 3, and using expression (10), the joint density function of the r-th and s-th concomitants is obtained from

$$\begin{aligned} f_{R_{[r]},R_{[s]}}(y_1,y_2) = {} & \frac{\lambda^2 e^{\mu_2(y_1+y_2)}}{\Gamma(r)\,\Gamma(s-r)} \left(\frac{\lambda}{\mu_1}\right)^{s-2} \int_0^\infty \int_0^{x_2} e^{\mu_1(x_1+x_2)} \left(e^{\mu_1 x_1}-1\right)^r \left(e^{\mu_1 x_2}-1\right) \\ & \times \exp\left[-\left(e^{\mu_1 x_2}-1\right)\left(\frac{\lambda}{\mu_1}+\frac{\left(e^{\mu_2 y_2}-1\right)}{\mu_2}\right)\right] \exp\left[-\left(e^{\mu_1 x_1}-1\right)\left(\frac{\left(e^{\mu_2 y_1}-1\right)}{\mu_2}\right)\right] \\ & \times \left[\left(e^{\mu_1 x_2}-1\right)-\left(e^{\mu_1 x_1}-1\right)\right]^{s-r-1} dx_1 dx_2 \end{aligned}$$

which, after the transformation $(e^{\mu_1 x_1}-1)=u$, $\mu_1 e^{\mu_1 x_1}dx_1=du$, turns out to be

$$f_{R_{[r]},R_{[s]}}(y_1,y_2)=\frac{\lambda^2 e^{\mu_2(y_1+y_2)}}{\mu_1\Gamma(r)\Gamma(s-r)}\left(\frac{\lambda}{\mu_1}\right)^{s-2}\int_0^\infty e^{\mu_1x_2}\left(e^{\mu_1x_2}-1\right)\exp\left[-\left(e^{\mu_1x_2}-1\right)\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2y_2}-1)}{\mu_2}\right)\right]$$
$$\times\left[\int_0^{e^{\mu_1x_2}-1}u^r\exp\left[-(u)\left(\frac{(e^{\mu_2y_1}-1)}{\mu_2}\right)\right]\left[\left(e^{\mu_1x_2}-1\right)-u\right]^{s-r-1}du\right]dx_2$$

and then

$$f_{R_{[r]},R_{[s]}}(y_1,y_2)=\frac{\lambda^2 e^{\mu_2(y_1+y_2)}}{\mu_1\Gamma(r)\Gamma(s-r)}\left(\frac{\lambda}{\mu_1}\right)^{s-2}\int_0^\infty e^{\mu_1x_2}\left(e^{\mu_1x_2}-1\right)$$
$$\times\exp\left[-\left(e^{\mu_1x_2}-1\right)\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2y_2}-1)}{\mu_2}\right)\right]I\left(e^{\mu_1x_2}-1\right)dx_2, \tag{11}$$

where

$$I\left(e^{\mu_1x_2}-1\right)=\int_0^{e^{\mu_1x_2}-1}u^r\exp\left[-(u)\left(\frac{(e^{\mu_2y_1}-1)}{\mu_2}\right)\right]\left[\left(e^{\mu_1x_2}-1\right)-u\right]^{s-r-1}du. \tag{12}$$

This integral expression is solved below by using a result in [18, Eq. 3.383-1, p. 347];

$$\int_0^u x^{v-1}(u-x)^{\mu-1}e^{\beta x}dx=B(\mu,v)\,u^{\mu+v-1}{}_1F_1(v,\mu+v;\beta u),\quad \mu>0,\ v>0$$

that, after the simplifications $u=(e^{\mu_1x_2}-1)$, $v=r+1$, $\mu=s-r$ and $\beta=-\frac{(e^{\mu_2y_2}-1)}{\mu_2}$, reduces to

$$I\left(e^{\mu_1x_2}-1\right)=\frac{\Gamma(r+1)\Gamma(s-r)}{\Gamma(s+1)}\left(e^{\mu_1x_2}-1\right)^s{}_1F_1\left(r+1,s+1;-\left(e^{\mu_1x_2}-1\right)\left(\frac{(e^{\mu_2y_1}-1)}{\mu_2}\right)\right), \tag{13}$$

where ${}_1F_1(a,b;x)$ is a Kummer confluent hypergeometric function given by Gradshteyn and Ryzhik [18] in the general form as ${}_1F_1(a,b;x)=\sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{j!}x^j$. Using these results, the joint density function now has the form

$$f_{R_{[r]},R_{[s]}}(y_1,y_2)=\frac{r\lambda^2 e^{\mu_2(y_1+y_2)}}{\mu_1\Gamma(s+1)}\left(\frac{\lambda}{\mu_1}\right)^{s-2}\int_0^\infty e^{\mu_1x_2}\left(e^{\mu_1x_2}-1\right)^{s+1}\exp\left[-\left(e^{\mu_1x_2}-1\right)\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2y_2}-1)}{\mu_2}\right)\right]$$
$$\times{}_1F_1\left(r+1,s+1;-\left(e^{\mu_1x_2}-1\right)\left(\frac{(e^{\mu_2y_1}-1)}{\mu_2}\right)\right)dx_2$$

that, after the transformation $(e^{\mu_1x_2}-1)=u$, $\mu_1e^{\mu_1x_2}dx_2=du$, becomes

$$=\frac{r\lambda^2 e^{\mu_2(y_1+y_2)}}{\mu_1^2\Gamma(s+1)}\left(\frac{\lambda}{\mu_1}\right)^{s-2}\int_0^\infty u^{s+1}\exp\left[-u\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2y_2}-1)}{\mu_2}\right)\right]{}_1F_1$$
$$\times\left(r+1,s+1;-u\left(\frac{(e^{\mu_2y_1}-1)}{\mu_2}\right)\right)du. \tag{14}$$

The solution of the integral expression above follows below;

$$\int_0^\infty u^{s+1}\exp\left[-u\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2y_2}-1)}{\mu_2}\right)\right]{}_1F_1\left(r+1,s+1;-u\frac{(e^{\mu_2y_1}-1)}{\mu_2}\right)du$$
$$=\Gamma(s+2)\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2y_2}-1)}{\mu_2}\right)^{s+1}{}_2F_1\left(r+1,s+2,s+1;\frac{-\frac{(e^{\mu_2y_1}-1)}{\mu_2}}{\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2y_2}-1)}{\mu_2}\right)}\right). \tag{15}$$

This integral expression can be computed by using a result in [18, Eq. 7.522-9, p. 815]

$$\int_0^\infty x^{\sigma-1}e^{-\beta u}\times{}_mF_n(a_1,\ldots,a_m;Q_1,\ldots,Q_n;\delta x)\,dx=\Gamma(\sigma)\,\beta^{-\sigma}{}_{m+1}F_n\left(a_1,\ldots,a_m,\sigma;Q_1,\ldots,Q_n;\frac{\delta}{\beta}\right),$$

$m\le n$, $\sigma>0$, $\beta>0$, if $m<n$; $\beta>\delta$, if $m=n$. To extend this to our case, let $\beta=\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2y_2}-1)}{\mu_2}\right)$ and $\delta=\frac{(e^{\mu_2y_1}-1)}{\mu_2}$ and consider

$$\int_0^\infty x^{s+1}e^{-\beta u}{}_1F_1(r+1,s+1;-\delta x)\,dx=\Gamma(s+1)\,\beta^{-s-1+r}(\beta+\delta)^{-r-2}(\beta(s+1)+\delta(s-r)),$$

$\beta, \delta > 0, s > -2$. Then, the following solution is obtained [24]

$$\int_0^\infty u^{s+1} \exp\left[-u\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2 y_2}-1)}{\mu_2}\right)\right] {}_1F_1\left(r+1, s+1; -u\frac{(e^{\mu_2 y_1}-1)}{\mu_2}\right) du$$
$$= \Gamma(s+1)\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2 y_2}-1)}{\mu_2}\right)^{-s-1+r}\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2 y_2}-1)}{\mu_2}+\frac{(e^{\mu_2 y_1}-1)}{\mu_2}\right)^{-r-2}$$
$$\times\left(\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2 y_2}-1)}{\mu_2}\right)(s+1)+\frac{(e^{\mu_2 y_1}-1)}{\mu_2}(s-r)\right). \quad (16)$$

The result in (16) leads us to the sought joint density function of the paired concomitants of the r-th and s-th records in explicit form:

$$f_{R_{[r]},R_{[s]}}(y_1,y_2) = \frac{r\lambda^2 e^{\mu_2(y_1+y_2)}}{\mu_1^2}\left(\frac{\lambda}{\mu_1}\right)^{s-2}\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2 y_2}-1)}{\mu_2}\right)^{-s-1+r}\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2 y_2}-1)}{\mu_2}+\frac{(e^{\mu_2 y_1}-1)}{\mu_2}\right)^{-r-2}$$
$$\times\left(\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2 y_2}-1)}{\mu_2}\right)(s+1)+\frac{(e^{\mu_2 y_1}-1)}{\mu_2}(s-r)\right) \quad (17)$$

or

$$f_{R_{[r]},R_{[s]}}(y_1,y_2) = r(s+2)e^{\mu_2(y_1+y_2)}\left(\frac{\lambda}{\mu_1}\right)^{s}\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2 y_2}-1)}{\mu_2}\right)^{s+1} {}_2F_1\left(r+1, s+2, s+1; \frac{-\frac{(e^{\mu_2 y_1}-1)}{\mu_2}}{\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2 y_2}-1)}{\mu_2}\right)}\right)$$

in terms of the components of expression (15).

6. Conclusion

The record values from a sequence of observations indicate the important features of extremities about the populations that they are observed from. These extremities with certain orders can even be coupled with some other variables which are in the capacity of their concomitants so that some analysis in larger scopes can be conducted. This paper achieves to show a bivariate model for such type of analysis. To this aim, this paper introduces the bivariate pseudo-Gompertz distribution and its upper records that are in the pairwise association with their concomitants. This sort of association is specified here with an exponential function. It is possible to increase the complexity of this function by adding in it new parameters or even to introduce some completely different functional forms as long as they are conceptually acceptable, intuitive and empirically identifiable for the age specific survival and hazard analysis. The proposed bivariate pseudo-Gompertz distribution has the complacency for the survival and hazard modeling and their extensions to the applications besides reliability analysis and actuarial modeling where age bound lifetime durations are of concern.

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