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Elastic Sturmian spirals in the Lorentz-Minkowski plane

Abstract: In this paper we consider some elastic spacelike and timelike curves in the Lorentz-Minkowski plane and obtain the respective vectorial equations of their position vectors in explicit analytical form. We study in more details the generalized Sturmian spirals in the Lorentz-Minkowski plane which simultaneously are elastics in this space.

Keywords: Elastic curves, Lorentz-Minkowski plane, Sturmian spirals

MSC: 53A35, 53B30, 53B50

1 Introduction

The curvature κ, of a given curve $\gamma : I \rightarrow M^n$ in a Riemannian manifold, can be interpreted as the tension that γ receives at each point as a result of the way it is immersed in the surrounding space. In 1740, Bernoulli proposed a simple geometric model for an elastic curve in $\mathbb{E}^2$, according to which an elastic curve or elastica is a critical point of the elastic energy functional $\int_\gamma \kappa^2$. Elastic curves in $\mathbb{E}^2$ were already classified by Euler in 1743 but it was not until 1928 that they were also studied in $\mathbb{E}^3$ by Radon, who derived the Euler-Lagrange equations and showed that they can be integrated explicitly. The elastica problem in real space forms has been recently considered using different approaches (see [1–5] and [6]).

Are there other interesting elastic curves? This question has been answered affirmatively by Marinov *et al.* in [7] where the Sturmian spirals in the Euclidean plane were described explicitly. There they have studied also the curves which belong to the class of the so called generalized Sturmian spirals which obey to the elastica equation. Eventually, they found analytical formulas for their parameterizations and presented a few illustrative plots.

Here, we are interested in elastic spacelike and timelike curves with tension in the Lorentz-Minkowski plane and derive the explicit vectorial equations of the respective generalized Sturmian spirals which belong to this class. By going to the three-dimensional pseudo-Euclidean space one can also study the third class of elastic Sturmian spirals on the lightlike cone in the spirit of [8], but this task is beyond the present study. The description of the generalized Sturmian spirals in the Lorentz-Minkowski plane can be found in [9].

2 Preliminaries

The Lorentz-Minkowski plane $\mathbb{E}_1^2$ is the Euclidean plane $\mathbb{R}^2$ equipped with an indefinite flat metric g given by the infinitesimal distance

$$\mathrm{d}s^2 = -\mathrm{d}x_1^2 + \mathrm{d}x_2^2$$

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where (x_1, x_2) are the rectangular coordinates in $\mathbb{E}_1^2$. Recall that any given vector $v \in \mathbb{E}_1^2 \backslash \{0\}$ can be either spacelike if $g(v, v) > 0$, timelike if $g(v, v) < 0$ or null (lightlike) if $g(v, v) = 0$. The norm of a vector v is provided by the formula $||v|| = \sqrt{|g(v, v)|}$. Two vectors v and w are said to be orthogonal, if $g(v, w) = 0$. An arbitrary curve $\gamma(s)$ in $\mathbb{E}_1^2$, can locally be spacelike or timelike, if all its velocity vectors $\gamma'(s)$ are, respectively, spacelike or timelike. Any spacelike or timelike curve γ can be parametrized by the so called arc-length parameter s for which $g(\gamma'(s), \gamma'(s)) = \pm 1$ (see [10]). In particular, every spacelike curve $\alpha(s)$ in the Lorentz-Minkowski plane can be represented in the form (cf. [11, 12] and [13])

$$\alpha(s) = \left(\int_0^s \sinh \varphi(s) \mathrm{d}s, \int_0^s \cosh \varphi(s) \mathrm{d}s \right) \tag{1}$$

and the corresponding Frenet vector fields along it are given by the formulas

$$\mathbf{T}(s) = (\sinh \varphi(s), \cosh \varphi(s)), \qquad \mathbf{N}(s) = (\cosh \varphi(s), \sinh \varphi(s)). \tag{2}$$

It is an easy task to see that they obey to the relations

$$\frac{\mathrm{d}\alpha}{\mathrm{d}s} = \mathbf{T}, \qquad \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \kappa \mathbf{N}, \qquad \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} = \kappa \mathbf{T} \tag{3}$$

in which the function

$$\kappa(s) = -g \left(\frac{\mathrm{d}\mathbf{T}(s)}{\mathrm{d}s}, \mathbf{N}(s) \right) = \frac{\mathrm{d}\varphi(s)}{\mathrm{d}s} \tag{4}$$

is the curvature of the curve.

The intrinsic equation of the spacelike elastic curves in the Lorentz-Minkowski plane reads

$$2\ddot{\kappa}(s) - \kappa^3(s) + \lambda \kappa(s) = 0 \tag{5}$$

in which the overdots mean derivatives with respect to s and λ is the tension constant.

Similarly, any timelike curve $\beta(s)$ in the Lorentz-Minkowski plane can be parameterized as follows

$$\beta(s) = \left(\int_0^s \cosh \phi(s) \mathrm{d}s, \int_0^s \sinh \phi(s) \mathrm{d}s \right). \tag{6}$$

This time the Frenet vector fields

$$\mathbf{T}(s) = (\cosh \phi(s), \sinh \phi(s)), \qquad \mathbf{N}(s) = (\sinh \phi(s), \cosh \phi(s)) \tag{7}$$

and the curvature

$$\kappa(s) = g \left(\frac{\mathrm{d}\mathbf{T}(s)}{\mathrm{d}s}, \mathbf{N}(s) \right) = \frac{\mathrm{d}\phi(s)}{\mathrm{d}s}$$

satisfy the relations

$$\frac{\mathrm{d}\beta}{\mathrm{d}s} = \mathbf{T}, \qquad \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \kappa \mathbf{N}, \qquad \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} = \kappa \mathbf{T}. \tag{8}$$

The respective intrinsic equation of the timelike elastic curves is

$$2\ddot{\kappa}(s) - \kappa^3(s) - \lambda \kappa(s) = 0 \tag{9}$$

with the same notation.

3 Elastic curves in the Lorentz-Minkowski plane

Below we will find the explicit parameterizations (up to quadratures) of the spacelike and timelike elastic curves in the Lorentz-Minkowski plane.

Theorem 3.1. *Let $\alpha : I \to \mathbb{E}_1^2$ be a spacelike elastic curve in the Lorentz-Minkowski plane. Then its position vector $\mathbf{x}(s) = (x(s), z(s))$ is given by the formulas*

$$x(s) = \frac{1}{c_2^2 - c_1^2} \left(\frac{c_2}{2} \int \kappa^2(s)\, ds + c_1 \kappa(s) - \frac{c_2}{2} \lambda s \right)$$

$$z(s) = \frac{1}{c_2^2 - c_1^2} \left(\frac{c_1}{2} \int \kappa^2(s)\, ds + c_2 \kappa(s) - \frac{c_1}{2} \lambda s \right)$$

(10)

where λ is the tension and the constants $c_1, c_2 \in \mathbb{R}$ are such that $c_1^2 \neq c_2^2$.

Proof. Following [14] we rewrite equation (5) into its equivalent form

$$\frac{d}{ds} \left(2\dot{\kappa}\mathbf{T} - \left(\kappa^2 - \lambda \right) \mathbf{N} \right) = 0.$$

(11)

From this equation it is easy to conclude that one has also the equation

$$2\dot{\kappa}\mathbf{T} - \left(\kappa^2 - \lambda \right) \mathbf{N} = \mathbf{C}_0$$

(12)

in which $\mathbf{C}_0 = (2c_1, 2c_2)$ is a constant vector. Substituting (2) in (12) we get

$$\dot{\kappa} \sinh(\varphi(s)) - \frac{1}{2}\left(\kappa^2 - \lambda \right) \cosh(\varphi(s)) = c_1$$

$$\dot{\kappa} \cosh(\varphi(s)) - \frac{1}{2}\left(\kappa^2 - \lambda \right) \sinh(\varphi(s)) = c_2.$$

(13)

Using (13) we obtain immediately the relations

$$\cosh(\varphi(s)) = \frac{1}{c_2^2 - c_1^2} \left(c_2 \dot{\kappa} + \frac{c_1}{2}\left(\kappa^2 - \lambda \right) \right)$$

$$\sinh(\varphi(s)) = \frac{1}{c_2^2 - c_1^2} \left(c_1 \dot{\kappa} + \frac{c_2}{2}\left(\kappa^2 - \lambda \right) \right)$$

(14)

provided that the constants $c_1, c_2 \in \mathbb{R}$ are such that $c_1^2 \neq c_2^2$. Substituting (14) in formula (1) we obtain exactly the parameterization given in (10). □

Theorem 3.2. *Let $\beta : I \to \mathbb{E}_1^2$ be a timelike elastic curve in Lorentz-Minkowski plane. Then the position vector $\mathbf{x}(s) = (x(s), z(s))$ is given by the formulas*

$$x(s) = \frac{1}{c_1^2 - c_2^2} \left(\frac{c_2}{2} \int \kappa^2(s)\, ds + c_1 \kappa(s) + \frac{c_2}{2} \lambda s \right)$$

$$z(s) = \frac{1}{c_1^2 - c_2^2} \left(\frac{c_1}{2} \int \kappa^2(s)\, ds + c_2 \kappa(s) + \frac{c_1}{2} \lambda s \right)$$

(15)

in which λ is the tension and the constants $c_1, c_2 \in \mathbb{R}$ are such that $c_1^2 \neq c_2^2$.

Proof. The proof of this theorem parallels that of the previous one, and will note only the differences. E.g., the analogue of (12) is

$$2\dot{\kappa}\mathbf{T} - \left(\kappa^2 + \lambda \right) \mathbf{N} = \mathbf{C}_0.$$

(16)

Respectively, equations (13) should be replaced with

$$\dot{\kappa} \cosh(\varphi(s)) - \frac{1}{2}\left(\kappa^2 + \lambda \right) \sinh(\varphi(s)) = c_1$$

$$\dot{\kappa} \sinh(\varphi(s)) - \frac{1}{2}\left(\kappa^2 + \lambda \right) \cosh(\varphi(s)) = c_2.$$

(17)

while equations (14) are transformed into

$$\cosh(\varphi(s)) = \frac{1}{c_1^2 - c_2^2}\left(c_1\dot{\kappa} + \frac{c_2}{2}\left(\kappa^2 + \lambda\right)\right)$$

$$\sinh(\varphi(s)) = \frac{1}{c_1^2 - c_2^2}\left(c_2\dot{\kappa} + \frac{c_1}{2}\left(\kappa^2 + \lambda\right)\right). \tag{18}$$

It is a matter of simple calculations to use the above to show the validity of Theorem 3.2.   □

4 Spacelike elastic Sturmian spirals

Let $\alpha : I \to \mathbb{E}_1^2$, $\alpha(s) = (x(s), z(s))$ be a spacelike elastic Sturmian spiral in Lorentz-Minkowski plane which means that its curvature is given by the function $\kappa = \sigma/r$, where $r = \sqrt{z^2 - x^2}$ and $\sigma \in \mathbb{R}^+$. Before making use of the intrinsic equation (5) let us integrate it with respect to s. This produces the equation

$$\dot{\kappa}^2 = \frac{1}{4}\kappa^4 - \frac{\lambda}{2}\kappa^2 + 2E \tag{19}$$

in which E denotes the integration constant which can be interpreted as energy. For the special case of the Sturmian spirals it can be rewritten in the form

$$\dot{r}^2 = \frac{\sigma^2}{4} - \frac{\lambda}{2}r^2 + \frac{2E}{\sigma^2}r^4. \tag{20}$$

As the tension λ is a physical property it can be assumed to be a positive constant. Therefore, there are only two cases to be considered, depending on the sign of E.

Case 1. Let us start by assuming that both E and λ are positive. We will write this fact formally as

$$E = \frac{a^2 c^2}{8} \qquad \text{and} \qquad \lambda = \frac{a^2 + c^2}{2}$$

with a and c being non-zero positive constants which comply with the condition $a > c$. Substituting the above expressions of E and λ in (20) we obtain

$$\dot{r}^2 = \frac{a^2 c^2}{4\sigma^2}\left(\frac{\sigma^2}{a^2} - r^2\right)\left(\frac{\sigma^2}{c^2} - r^2\right). \tag{21}$$

As a consequence, the solutions of the differential equation (21) are either

$$r(s) = \frac{\sigma}{a}\text{sn}\left(\frac{as}{2}, k\right), \qquad k = \frac{c}{a}, \qquad r < \frac{\sigma}{a} \tag{22}$$

or

$$r(s) = \frac{\sigma}{c}\frac{1}{\text{sn}\left(\frac{as}{2}, k\right)}, \qquad k = \frac{c}{a}, \qquad r > \frac{\sigma}{c} \tag{23}$$

where the first slot in the Jacobian sinus elliptic function $\text{sn}(\cdot, \cdot)$ is reserved for the argument, the second one for the so called elliptic modulus which is a real number between zero and one.

Now, when (22) holds we have

$$\kappa(s) = \frac{\sigma}{r(s)} = \frac{a}{\text{sn}\left(\frac{as}{2}, k\right)}, \qquad \dot{\kappa}(s_0) = 0. \tag{24}$$

The equality on the right hand side is satisfied for $s_0 = 2K(k)/a$, where $K(k)$ is the complete elliptic integral of the first kind and $\kappa(s_0) = a$. Then using (24) in (13) we can conclude that $(c_1, c_2) = \left(\left(c^2 - a^2\right)/4, 0\right)$. Taking into account all above and Theorem 3.1 we end up with the parameterization

$$x(s) = \frac{4a}{a^2 - c^2}\frac{1}{\text{sn}\left(\frac{as}{2}, k\right)}, \qquad k = c/a$$

$$z(s) = \frac{4a}{a^2 - c^2} F\left(\mathrm{am}\left(\frac{as}{2}, k\right), k\right) - \frac{4a}{a^2 - c^2} E\left(\mathrm{am}\left(\frac{as}{2}, k\right), k\right)$$
$$- \frac{4a}{a^2 - c^2} \frac{\mathrm{cn}\left(\frac{as}{2}, k\right) \mathrm{dn}\left(\frac{as}{2}, k\right)}{\mathrm{sn}\left(\frac{as}{2}, k\right)} - \frac{a^2 + c^2}{a^2 - c^2} s \tag{25}$$

where $\mathrm{am}(\cdot, \cdot)$ is the Jacobian amplitude function, cn and dn are the remaining Jacobian elliptic functions and $F(\cdot, \cdot)$, $E(\cdot, \cdot)$ denote the incomplete elliptic inegrals of the first, respectively second kind (for more details see [15]).

Now, let us switch to the solution presented in (23). Proceeding in the same manner we obtain

$$\kappa(s) = \frac{\sigma}{r} = c\left(\mathrm{sn}\left(\frac{as}{2}, k\right)\right) \qquad \text{and} \qquad \dot{\kappa}(s_0) = 0 \tag{26}$$

for $s_0 = 2\left(K(k)\right)/a$ complemented by $\kappa(s_0) = c$. Entering with (26) in (13) we obtain that $(c_1, c_2) = \left((a^2 - c^2)/4, 0\right)$. Substituting (26) in the formulas of Theorem 1, we find

$$x(s) = \frac{4c}{c^2 - a^2} \mathrm{sn}\left(\frac{as}{2}, k\right), \qquad k = c/a$$
$$z(s) = \frac{4a}{c^2 - a^2}\left(F\left(\mathrm{am}\left(\frac{as}{2}, k\right), k\right) - E\left(\mathrm{am}\left(\frac{as}{2}, k\right), k\right)\right) - \frac{a^2 + c^2}{c^2 - a^2} s. \tag{27}$$

Case 2. When E is negative, we can write respectively

$$E = -\frac{a^2 c^2}{8} \qquad \text{and} \qquad \lambda = \frac{a^2 - c^2}{2}. \tag{28}$$

Inserting E and λ in (20) we obtain the equation

$$\dot{r}^2 = \frac{a^2 c^2}{4\sigma^2}\left(\frac{\sigma^2}{a^2} - r^2\right)\left(\frac{\sigma^2}{c^2} + r^2\right). \tag{29}$$

The solution of the above differential equation is

$$r(s) = \frac{\sigma}{a}\mathrm{cn}(\frac{cs}{2k}, k), \qquad k = \frac{c}{\sqrt{a^2 + c^2}}, \qquad r < \frac{\sigma}{a} \tag{30}$$

and therefore

$$\kappa(s) = \frac{\sigma}{r(s)} = \frac{a}{\mathrm{cn}(\frac{cs}{2k}, k)}, \qquad \dot{\kappa}(0) = 0 \quad \text{and} \quad \kappa(0) = a. \tag{31}$$

All of this, along with (13), produces $(c_1, c_2) = \left(-(a^2 + c^2)/4, 0\right)$. Relying on Theorem 3.1 we obtain finally the parameterization of the curve, i.e.,

$$x(s) = \frac{4a}{a^2 + c^2} \frac{1}{\mathrm{cn}(\frac{cs}{2k}, k)}, \qquad k = \frac{c}{\sqrt{a^2 + c^2}}$$
$$z(s) = \frac{4ka^2}{c(a^2 + c^2)} F\left(\mathrm{am}\left(\frac{cs}{2k}, k\right), k\right) - \frac{4k}{c} E\left(\mathrm{am}\left(\frac{cs}{2k}, k\right), k\right)$$
$$- \frac{4k}{c} \frac{\mathrm{cn}\left(\frac{cs}{2k}, k\right) \mathrm{dn}\left(\frac{cs}{2k}, k\right)}{\mathrm{sn}\left(\frac{cs}{2k}, k\right)} - \frac{a^2 - c^2}{a^2 + c^2} s. \tag{32}$$

5 Timelike elastic Sturmian spirals

Let $\beta : I \to \mathbb{E}_1^2$, $\beta(s) = (x(s), z(s))$ be a timelike elastic Sturmian spiral in the Lorentz-Minkowski plane. Integrating the intrinsic equation (9) we have

$$\dot{\kappa}^2 = \frac{1}{4}\kappa^4 + \frac{\lambda}{2}\kappa^2 + 2E \tag{33}$$

Fig. 1. The left figure is obtained by formulas (25) with $a = 2$ and $c = 1$, this one in the middle via formulas (27) with $a = 3$ and $c = 1$, and that one at most right by formulas (32) with $a = 1$ and $c = 2$. In all these cases $\sigma \equiv 2$.
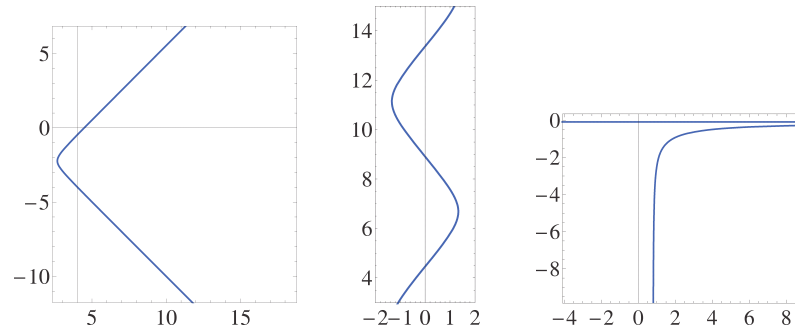
where E as before is the constant of integration (the energy). Since β is a Sturmian spiral, the curvature function κ is given by the function $\kappa = \dfrac{\sigma}{r}$, where $r = \sqrt{x^2 - z^2}$ and $\sigma \in \mathbb{R}^+$. Rewriting appropriately the equation (33), we find

$$\dot{r}^2 = \frac{\sigma^2}{4} + \frac{\lambda}{2}r^2 + \frac{2E}{\sigma^2}r^4. \tag{34}$$

Again, depending on the sign of E there are two cases.

Case 1. Both E and λ are positive which is ensured by writing

$$E = \frac{a^2 c^2}{8} \quad \text{and} \quad \lambda = \frac{a^2 + c^2}{2}.$$

Substituting these values of E and λ in (34), we obtain

$$\dot{r}^2 = \frac{a^2 c^2}{4\sigma^2}\left(\frac{\sigma^2}{a^2} + r^2\right)\left(\frac{\sigma^2}{c^2} + r^2\right). \tag{35}$$

The solution of the above differential equation is of the form

$$r\,(s) = \frac{\sigma}{c}\frac{\mathrm{cn}\left(\frac{as}{2}, k\right)}{\mathrm{sn}\left(\frac{as}{2}, k\right)}, \qquad k = \frac{\sqrt{a^2 - c^2}}{a}. \tag{36}$$

Therefore we get

$$\kappa\,(s) = \frac{\sigma}{r\,(s)} = c\,\frac{\mathrm{sn}\left(\frac{as}{2}, k\right)}{\mathrm{cn}\left(\frac{as}{2}, k\right)}, \qquad (c_1, c_2) = (\frac{ac}{2}, -\frac{a^2 + c^2}{4}). \tag{37}$$

Substituting (37) in Theorem 3.2, we find

$$
\begin{aligned}
x\,(s) = {} & \frac{4a\,(a^2 + c^2)}{(a^2 - c^2)^2}\left(\frac{\mathrm{dn}\left(\frac{as}{2}, k\right)\mathrm{sn}\left(\frac{as}{2}, k\right)}{\mathrm{cn}\left(\frac{as}{2}, k\right)} - E\left(\mathrm{am}\left(\frac{as}{2}, k\right), k\right)\right) \\
& - \frac{8ac^2}{(a^2 - c^2)^2}\frac{\mathrm{sn}\left(\frac{as}{2}, k\right)}{\mathrm{cn}\left(\frac{as}{2}, k\right)} + \frac{(a^2 + c^2)^2}{(a^2 - c^2)^2}s \\
z\,(s) = {} & -\frac{8a^2 c}{(a^2 - c^2)^2}\left(\frac{\mathrm{dn}\left(\frac{as}{2}, k\right)\mathrm{sn}\left(\frac{as}{2}, k\right)}{\mathrm{cn}\left(\frac{as}{2}, k\right)} - E\left(\mathrm{am}\left(\frac{as}{2}, k\right), k\right)\right) \\
& + \frac{4c\,(a^2 + c^2)}{(a^2 - c^2)^2}\frac{\mathrm{sn}\left(\frac{as}{2}, k\right)}{\mathrm{cn}\left(\frac{as}{2}, k\right)} - \frac{2ac\,(a^2 + c^2)}{(a^2 - c^2)^2}s.
\end{aligned}
\tag{38}
$$

Case 2. Assuming that E is negative, that is

$$E = -\frac{a^2 c^2}{8} \quad \text{and} \quad \lambda = \frac{a^2 - c^2}{2} \tag{39}$$

we find that the intrinsic equation of the sought curves is

$$\dot{r}^2 = \frac{a^2 c^2}{4\sigma^2}\left(\frac{\sigma^2}{a^2} + r^2\right)\left(\frac{\sigma^2}{c^2} - r^2\right). \tag{40}$$

Regarding the solution of the above differential equation we have

$$r(s) = \frac{\sigma}{c}\mathrm{cn}\left(\frac{as}{2k}, k\right), \qquad k = \frac{a}{\sqrt{a^2 + c^2}}, \qquad r < \frac{\sigma}{c}. \tag{41}$$

From (41) we have also

$$\kappa = \frac{\sigma}{r(s)} = \frac{c}{\mathrm{cn}\left(\frac{as}{2k}, k\right)}, \qquad \kappa'(0) = 0 \qquad \text{and} \qquad \kappa(0) = c. \tag{42}$$

A similar method as in the spacelike case produces $(c_1, c_2) = \left(0, -\left(a^2 + c^2\right)/4\right)$. By using (42) in Theorem 3.2 we have finally the explicit parameterization

$$x(s) = \frac{4a^2}{\left(a^2 + c^2\right)^{\frac{3}{2}}} F\left(\mathrm{am}\left(\frac{cs}{2k}, k\right), k\right) - \frac{4}{\sqrt{a^2 + c^2}} E\left(\mathrm{am}\left(\frac{cs}{2k}, k\right), k\right)$$

$$\qquad\qquad - \frac{4}{\sqrt{a^2 + c^2}}\frac{\mathrm{dn}\left(\frac{cs}{2k}, k\right)\mathrm{cn}\left(\frac{cs}{2k}, k\right)}{\mathrm{sn}\left(\frac{cs}{2k}, k\right)} + \frac{a^2 - c^2}{a^2 + c^2}s \tag{43}$$

$$z(s) = \frac{4c}{a^2 + c^2}\frac{1}{\mathrm{cn}\left(\frac{as}{2k}, k\right)}.$$

Both cases of timelike elastic Sturmian spirals in the Lorentz-Minkowski plane are pictured in Figure 2 for a specific choice of the parameters.

Fig. 2. The left hand side figure is obtained by formulas (38), and that one on the right via formulas (43). In both cases $a = 2$, $c = 1$ and $\sigma = 1$.
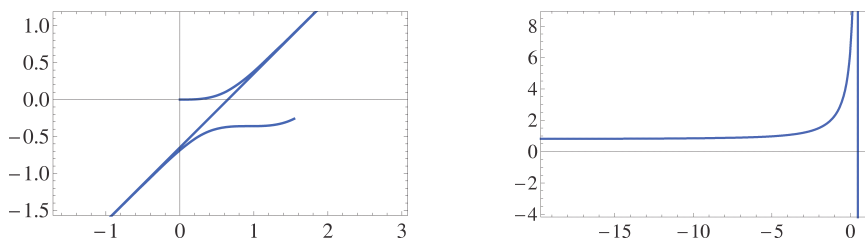
6 Conclusions

The Lorentz-Minkowski plane is equipped with an indefinite metric and one should expect different types of curves to those in the Euclidean case. Elsewhere Marinov *et al* [7] have studied the elastic Sturmian spirals in the Euclidean plane. By drawing inspiration from this work we have considered here the spacelike and timelike elastic Sturmian spirals in the Lorentz-Minkowski plane $\mathbb{E}_1^2$. Our main results are the derivation of the explicit parametric equations for spacelike and timelike elastic Sturmian spirals in $\mathbb{E}_1^2$. We have presented also in Figure 1 and Figure 2 some graphical illustrations of these curves that are realized using `Mathematica`®.

    Last but not least, we believe that the results presented in this paper suggest that some other curves like the elastic Serret's curves, elastic Bernoulli's Lemniscate and the generalized elastic curves whose curvature depend on the distance from the origin deserve to be studied in some details in the Lorentz-Minkowski plane $\mathbb{E}_1^2$ as well. We are planning to report on realization of this program elsewhere.

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