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ON TIMELIKE BERTRAND CURVES IN MINKOWSKI 3-SPACE

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Abstract. In this paper, we study the timelike Bertrand curves in Minkowski 3-space. Since the principal normal vector of a timelike curve is spacelike, the Bertrand mate curve of this curve can be a timelike curve, a spacelike curve with spacelike principal normal or a Cartan null curve, respectively. Thus, by considering these three cases, we get the necessary and sufficient conditions for a timelike curve to be a Bertrand curve. Also we give the related examples.

1. Introduction

A classical problem in Differential Geometry raised by Saint-Venant in 1845([14]) led to discovery of Bertrand curves in 1850 ([3]). A Bertrand curve is a curve in the Euclidean space such that its principal normal is the principal normal of the second curve. J. Bertrand proved that a necessary and sufficient condition for the existence of such a second curve is required in fact a linear relationship calculated with constant coefficients should exist between the first and second curvatures of the given original curve. In other words, if we denote first and second curvatures of a given curve by k_1 and k_2 respectively, we have $\lambda k_1 + \mu k_2 = 1$, $\lambda, \mu \in R$. Since 1850, after the paper of Bertrand, the pairs of curves like this have been called Conjugate Bertrand Curves, or more commonly Bertrand Curves (see [8]).

The study of this kind of curves has been extended to many other ambient spaces. In [10], Pears studied this problem for curves in the n-dimensional Euclidean space $\mathbb{E}^n$, $n > 3$, and showed that a Bertrand curve in $\mathbb{E}^n$ must belong to a three-dimensional subspace $\mathbb{E}^3 \subset \mathbb{E}^n$. This

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result is restated by Matsuda and Yorozu [9]. They proved that there was not any special Bertrand curves in $\mathbb{E}^n$ $(n > 3)$ and defined a new kind, which is called $(1,3)$-type Bertrand curves in 4-dimensional Euclidean space. Bertrand curves and their characterizations were studied by many researchers in Minkowski 3-space and Minkowski space-time (see [1], [2], [6], [7], [11]) as well as in Euclidean space. In addition, in [12] and [13], the authors studied $(1,3)$-type Bertrand curves in semi-Euclidean 4-space with index 2.

In the present paper, we study the timelike Bertrand curves in Minkowski 3-space. Since the principal normal vector of a timelike curve is spacelike, the Bertrand mate curve of this curve can be a timelike curve, a spacelike curve with spacelike principal normal or a Cartan null curve, respectively. Thus, by considering these three cases, we get the necessary and sufficient conditions for a timelike curve to be a Bertrand curve. Also we give the related examples.

2. Preliminaries

The Minkowski space $\mathbb{E}_1^3$ is the 3-dimensional real vector space $\mathbb{R}^3$ equipped with the indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{R}^3$. Recall that a vector $v \in \mathbb{E}_1^3 \backslash \{0\}$ can be *spacelike* if $g(v,v) > 0$, *timelike* if $g(v,v) < 0$ and *null (lightlike)* if $g(v,v) = 0$ and $v \neq 0$. In particular, the vector $v = 0$ is spacelike. The norm of a vector v is given by $||v|| = \sqrt{|g(v,v)|}$, and two vectors v and w are said to be orthogonal, if $g(v,w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}_1^3$, can locally be *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null ([8]). A spacelike curve in $\mathbb{E}_1^3$ is called *pseudo null curve* if its principal normal vector N is null [4]. A null curve α is said to be parameterized by pseudo-arc s if $g(\alpha''(s), \alpha''(s)) = 1$. A spacelike or a timelike curve α is said to be parameterized by arc-length s if $g(\alpha'(s), \alpha'(s)) = \pm 1$ ([4]).

Let $\{T, N, B\}$ be the moving Frenet frame along a curve α in $\mathbb{E}_1^3$, consisting of the tangent, the principal normal and the binormal vector fields, respectively. Depending on the causal character of α, the Frenet equations have the following forms.

Case I. If α is a non-null curve, the Frenet equations are given by ([8]):

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2 k_1 & 0 \\ -\epsilon_1 k_1 & 0 & \epsilon_3 k_2 \\ 0 & -\epsilon_2 k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{1}$$

where k_1 and k_2 are the first and the second curvature of the curve respectively. Moreover, the following conditions hold:

$$g(T,T) = \epsilon_1 = \pm 1, \quad g(N,N) = \epsilon_2 = \pm 1, \quad g(B,B) = \epsilon_3 = \pm 1$$

and

$$g(T,N) = g(T,B) = g(N,B) = 0.$$

Case II. If α is a null curve, the Frenet equations are given by ([4])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_2 & 0 & -k_1 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2}$$

where the first curvature $k_1 = 0$ if α is straight line, or $k_1 = 1$ in all other cases. In particular, the following conditions hold:

$$g(T,T) = g(B,B) = g(T,N) = g(N,B) = 0, \quad g(N,N) = g(T,B) = 1.$$

Case III. If α is a pseudo null curve, the Frenet formulas have the form ([5])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ 0 & k_2 & 0 \\ -k_1 & 0 & -k_2 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{3}$$

where the first curvature $k_1 = 0$ if α is straight line, or $k_1 = 1$ in all other cases. In particular, the following conditions hold:

$$g(N,N) = g(B,B) = g(T,N) = g(T,B) = 0, \quad g(T,T) = g(N,B) = 1.$$

3. Timelike Bertrand curves in Minkowski 3-space $\mathbb{E}_1^3$

In this section, we consider the timelike Bertrand curves in $\mathbb{E}_1^3$. We get the necessary and sufficient conditions for the timelike curves to be Bertrand curves in $\mathbb{E}_1^3$ and we also give the related examples.

Definition 3.1. *A timelike curve $\alpha : I \to \mathbb{E}^3_1$ with $\kappa_1(s) \neq 0$ is a Bertrand curve if there is a curve $\alpha^* : I^* \to \mathbb{E}^3_1$ such that the principal normal vectors of $\alpha(s)$ and $\alpha^*(s^*)$ at $s \in I$, $s^* \in I^*$ are equal. In this case, $\alpha^*(s^*)$ is called the Bertrand mate of $\alpha(s)$.*

Let $\beta : I \to \mathbb{E}^3_1$ be a timelike Bertrand curve in $\mathbb{E}^3_1$ with the Frenet frame $\{T, N, B\}$ and the curvatures κ_1, κ_2, and $\beta^* : I \to \mathbb{E}^3_1$ be a Bertrand mate curve of β with the Frenet frame $\{T^*, N^*, B^*\}$ and the curvatures κ^*_1, κ^*_2.

Theorem 3.2. *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}^3_1$ be a unit speed timelike curve with the non-zero curvatures κ_1, κ_2. Then the curve β is a Bertrand curve with Bertrand mate β^* if and only if one of the following conditions holds:*
(i) there exist constant real numbers λ and h satisfying

$$h^2 > 1, \quad 1 + \lambda\kappa_1 = h\lambda\kappa_2, \quad h\kappa_1 - \kappa_2 \neq 0, \quad h\kappa_2 - \kappa_1 \neq 0. \tag{4}$$

In this case, β^* is a timelike curve in $\mathbb{E}^3_1$.

(ii) there exist constant real numbers λ and h satisfying

$$h^2 < 1, \quad 1 + \lambda\kappa_1 = h\lambda\kappa_2, \quad h\kappa_1 - \kappa_2 \neq 0, \quad h\kappa_2 - \kappa_1 \neq 0. \tag{5}$$

In this case, β^* is a spacelike curve with spacelike principal normal in $\mathbb{E}^3_1$.

Proof. Assume that β is a timelike Bertrand curve parametrized by arc-length s with non-zero curvatures κ_1, κ_2 and the curve β^* is the Bertrand mate curve of the curve β parametrized by with arc-length or pseudo arc s^*.

(i) Let β^* be a timelike curve. The proof of this case can be similarly done to the theorem in [15].

(ii) Let β^* be a spacelike curve with spacelike principal normal. Then, we can write the curve β^* as

$$\beta^*(s^*) = \beta^*(f(s)) = \beta(s) + \lambda(s)N(s) \tag{6}$$

for all $s \in I$ where $\lambda(s)$ is $C^\infty-$function on I. Differentiating (6) with respect to s and using (1), we get

$$T^* f' = (1 + \lambda\kappa_1)T + \lambda' N + \lambda\kappa_2 B. \tag{7}$$

By taking the scalar product of (7) with N, we have

$$\lambda' = 0. \tag{8}$$

Substituting (8) in (7), we find

$$T^* f' = (1 + \lambda\kappa_1)T + \lambda\kappa_2 B. \tag{9}$$

By taking the scalar product of (9) with itself, we obtain

$$\left(f'\right)^2 = -(1 + \lambda\kappa_1)^2 + (\lambda\kappa_2)^2 . \tag{10}$$

If we denote

$$\delta = \frac{1 + \lambda\kappa_1}{f'} \quad \text{and} \quad \gamma = \frac{\lambda\kappa_2}{f'}, \tag{11}$$

we get

$$T^* = \delta T + \gamma B_1. \tag{12}$$

Differentiating (12) with respect to s and using (1), we find

$$f'\kappa_1^* N^* = \delta' T + \left(\delta\kappa_1 - \gamma\kappa_2\right) N + \gamma' B. \tag{13}$$

By taking the scalar product of (13) with itself, we get

$$\delta' = 0 \quad \text{and} \quad \gamma' = 0. \tag{14}$$

Since $\gamma \neq 0$, we have $1 + \lambda\kappa_1 = h\lambda\kappa_2$ where $h = \delta/\gamma$. Substituting (14) in (13) , we find

$$f'\kappa_1^* N^* = \left(\delta\kappa_1 - \gamma\kappa_2\right) N \tag{15}$$

By taking the scalar product of (15) with itself, using (10) and (11), we have

$$\left(f'\right)^2 \left(\kappa_1^*\right)^2 = \frac{\left(h\kappa_1 - \kappa_2\right)^2}{1 - h^2} \tag{16}$$

where $h\kappa_1 - \kappa_2 \neq 0$ and $h^2 < 1$. If we put $v = \left(\delta\kappa_1 - \gamma\kappa_2\right)/f'\kappa_1^*$, we get

$$N^* = vN. \tag{17}$$

Differentiating (17) with respect to s and using (1), we find

$$-f'\kappa_2^* B^* = v\kappa_1 T + v\kappa_2 B + f'\kappa_1^* T^* \tag{18}$$

where $v' = 0$. Rewriting (18) by using (9) , we get

$$-f'\kappa_2^* B^* = P\left(s\right) T + Q\left(s\right) B \tag{19}$$

where

$$\begin{aligned} P\left(s\right) &= \frac{\lambda\kappa_2\left(h\kappa_1 - \kappa_2\right)}{(f')^2\kappa_1^*\left(1 - h^2\right)}\left(\kappa_1 - h\kappa_2\right), \\ Q\left(s\right) &= \frac{\lambda\kappa_2\left(h\kappa_1 - \kappa_2\right)h}{(f')^2\kappa_1^*\left(1 - h^2\right)}\left(\kappa_1 - h\kappa_2\right) \end{aligned}$$

which implies that $h\kappa_2 - \kappa_1 \neq 0$.

Conversely, assume that β is a timelike curve parametrized by arc-length s with non-zero curvatures κ_1, κ_2, and the conditions of (4) hold for constant real numbers λ and h. Then, we can define a curve β^* as

$$\beta^*(s^*) = \beta(s) + \lambda N(s). \tag{20}$$

Differentiating (20) with respect to s and using (1), we find

$$\frac{d\beta^*}{ds} = \lambda\kappa_2\left(hT + B\right) \tag{21}$$

which leads to that

$$f' = \sqrt{\left|g\left(\frac{d\beta^*}{ds}, \frac{d\beta^*}{ds}\right)\right|} = m_1\lambda\kappa_2\sqrt{1-h^2} \tag{22}$$

where $m_1 = \pm 1$ such that $m_1\lambda\kappa_2 > 0$. Rewriting (21) , we obtain

$$T^* = \frac{m_1}{\sqrt{1-h^2}}\left(hT + B\right), \quad g\left(T^*, T^*\right) = 1. \tag{23}$$

Differentiating (23) with respect to s and using (1), we get

$$\frac{dT^*}{ds^*} = \frac{m_1\left(h\kappa_1 - \kappa_2\right)}{f'\sqrt{1-h^2}}N \tag{24}$$

which causes that

$$\kappa_1^* = \left\|\frac{dT^*}{ds^*}\right\| = \frac{m_2\left(h\kappa_1 - \kappa_2\right)}{f'\sqrt{1-h^2}} \tag{25}$$

where $m_2 = \pm 1$ such that $m_2\left(h\kappa_1 - \kappa_2\right) > 0$. Now, we can find N^* as

$$N^* = m_1 m_2 N, \quad g\left(N^*, N^*\right) = 1. \tag{26}$$

Differentiating (26) with respect to s, using (23) and (24), we get

$$\frac{dN^*}{ds^*} + \kappa_1^* T^* = \frac{m_1 m_2\left(\kappa_1 - h\kappa_2\right)}{f'\left(1-h^2\right)}\left(T + hB\right) \tag{27}$$

which bring about that

$$\kappa_2^* = \frac{m_3\left(\kappa_1 - h\kappa_2\right)}{f'\sqrt{1-h^2}},$$

where $m_3 = \pm 1$ such that $m_3\left(\kappa_1 - h\kappa_2\right) > 0$. Lastly, we define B^* as

$$B^* = \frac{m_1 m_2 m_3}{\sqrt{1-h^2}}\left(T + hB\right), \quad g\left(B^*, B^*\right) = -1.$$

Then β^* is a spacelike curve with spacelike principal normal and the Bertrand mate curve of β. Thus β is a Bertrand curve. □

Theorem 3.3. *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a unit speed timelike curve with the non-zero curvatures κ_1, κ_2 and $\beta^* : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a Cartan null curve with curvatures $\kappa_1^* = 1$, κ_2^*. If the curve β^* is a Bertrand mate curve of the curve β, then there exist constant real numbers λ and $h = \pm 1$ satisfying $1 + \lambda\kappa_1 = h\lambda\kappa_2$ and $h\kappa_1 - \kappa_2 \neq 0$.*

Proof. Assume that β is a timelike Bertrand curve parametrized by arc-length s with non-zero curvatures κ_1, κ_2 and the curve β^* is the Cartan null Bertrand mate curve of the curve β parametrized by with pseudo arc s^* with curvatures $\kappa_1^* = 1$, κ_2^*. Then, we can write the curve β^* as

$$\beta^*(s^*) = \beta^*(f(s)) = \beta(s) + \lambda(s)N(s) \tag{28}$$

for all $s \in I$ where $\lambda(s)$ is $C^\infty-$function on I. Using (1) and (2), differentiating (28) with respect to s, we get

$$T^* f' = (1 + \lambda\kappa_1)T + \lambda' N + \lambda\kappa_2 B. \tag{29}$$

By taking the scalar product of (29) with N, we have

$$\lambda' = 0. \tag{30}$$

Substituting (30) in (29), we find

$$T^* f' = (1 + \lambda\kappa_1)T + \lambda\kappa_2 B. \tag{31}$$

By taking the scalar product of (9) with itself, we obtain

$$(1 + \lambda\kappa_1)^2 = (\lambda\kappa_2)^2 \tag{32}$$

which implies that $1 + \lambda\kappa_1 = h\lambda\kappa_2$ where $h = \pm 1$. Rewriting (31) by using (32) , we get

$$T^* f' = \lambda\kappa_2(hT + B). \tag{33}$$

Putting $v = \lambda\kappa_2 / f'$ and differentiating (33) with respect to s by using (1), we find

$$f' N^* = a\,(h\kappa_1 - \kappa_2)\,N \tag{34}$$

which means that $h\kappa_1 - \kappa_2 \neq 0$. □

Theorem 3.4. *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a unit speed timelike curve with non-zero constant curvatures κ_1, κ_2 and $\beta^* : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a Cartan null curve with curvatures $\kappa_1^* = 1$, κ_2^*. Then the curve β^* is a Bertrand mate curve of the curve β if and only if there exist constant real numbers λ and $h = \pm 1$ satisfying $1 + \lambda\kappa_1 = h\lambda\kappa_2$ and $h\kappa_1 - \kappa_2 \neq 0$.*

Proof. Assume that β is a timelike Bertrand curve parametrized by arc-length s with non-zero constant curvatures κ_1, κ_2 and the curve β^* is the Cartan null Bertrand mate curve of the curve β parametrized by with pseudo arc s^* with curvatures $\kappa_1^* = 1$, κ_2^*. Then from above theorem, there exist constant real numbers λ and $h = \pm 1$ satisfying $1 + \lambda\kappa_1 = h\lambda\kappa_2$ and $h\kappa_1 - \kappa_2 \neq 0$.

Conversely, assume that β is a timelike curve parametrized by arc-length s with non-zero constant curvatures κ_1, κ_2 and there exist constant real numbers λ and $h = \pm 1$ satisfying $1 + \lambda\kappa_1 = h\lambda\kappa_2$ and $h\kappa_1 - \kappa_2 \neq 0$. Then, we can define a curve β^* as

$$\beta^*(s^*) = \beta(s) + \lambda N(s). \tag{35}$$

Differentiating (35) with respect to s and using (1), we find

$$\frac{d\beta^*}{ds} = \lambda\kappa_2 \left(hT + B\right). \tag{36}$$

Differentiating (35) with respect to s and using (1), we find

$$\frac{d^2\beta^*}{ds^2} = \lambda\kappa_2 \left(h\kappa_1 - \kappa_2\right) N \tag{37}$$

which leads to that

$$f' = \left(\left| g\left(\frac{d\beta^*}{ds}, \frac{d\beta^*}{ds}\right)\right|\right)^{1/4} = \sqrt{m_1\lambda\kappa_2\left(h\kappa_1 - \kappa_2\right)} \tag{38}$$

where $m_1 = \pm 1$ such that $m_1\lambda\kappa_2\left(h\kappa_1 - \kappa_2\right) > 0$. Rewriting (36) and (37), we obtain

$$T^* = \frac{\lambda\kappa_2}{\sqrt{m_1\lambda\kappa_2\left(h\kappa_1 - \kappa_2\right)}}\left(hT + B\right), \quad g\left(T^*, T^*\right) = 0, \tag{39}$$

$$N^* = m_1 N, \quad g\left(N^*, N^*\right) = 1 \quad \text{and} \quad \kappa_1^* = 1. \tag{40}$$

We know that $\kappa_2^* = -\frac{1}{2} g\left(\frac{dN^*}{ds^*}, \frac{dN^*}{ds^*}\right)$. Thus we have

$$\kappa_2^* = \frac{\kappa_1^2 - \kappa_2^2}{2m_1\lambda\kappa_2\left(h\kappa_1 - \kappa_2\right)}. \tag{41}$$

Lastly, we can define B^* as

$$B^* = \kappa_2^* T^* - \frac{dN^*}{ds^*} = \frac{-\lambda\kappa_2 h\left(h\kappa_1 - \kappa_2\right)^2}{2\left(m_1\lambda\kappa_2\left(h\kappa_1 - \kappa_2\right)\right)^{3/2}}\left(T - hB\right), \quad g\left(B^*, B^*\right) = 0.$$

Then β^* is a Cartan null curve and the Bertrand mate curve of β. Thus β is a Bertrand curve. □

Example 1. Let us consider a timelike curve in $\mathbb{E}_1^3$ with the equation

$$\beta(s) = \left(\sqrt{2}\sinh s, \sqrt{2}\cosh s, s\right)$$

with the Frenet Frame

$$\begin{aligned} T(s) &= \left(\sqrt{2}\cosh s, \sqrt{2}\sinh s, 1\right), \\ N(s) &= (\sinh s, \cosh s, 0), \\ B_1(s) &= \left(\cosh s, \sinh s, \sqrt{2}\right) \end{aligned}$$

and the curvatures $\kappa_1(s) = \sqrt{2}$ and $\kappa_2(s) = -1$. If we take $h = \sqrt{2}$ and $\lambda = -1/2\sqrt{2}$ in (i) of theorem 3.2, then we get the curve β^* as follows:

$$\beta^*(s) = \beta(s) - \frac{1}{2\sqrt{2}}N(s) = \left(\frac{3}{2\sqrt{2}}\sinh s, \frac{3}{2\sqrt{2}}\cosh s, s\right)$$

By straight calculations, we get

$$\begin{aligned} T^*(s) &= \left(3\cosh s, 3\sinh s, 2\sqrt{2}\right), \\ N^*(s) &= (\sinh s, \cosh s, 0), \\ B^*(s) &= \left(2\sqrt{2}\cosh s, 2\sqrt{2}\sinh s, 3\right) \end{aligned}$$

and $\kappa_1^*(s) = 6\sqrt{2}$, $\kappa_2^*(s) = -8$. It can be easily seen that the curve β^* is a timelike Bertrand mate curve of the curve β.

Example 2. For the same timelike curve β in Example 1, if we take $h = \sqrt{2}/2$ and $\lambda = -\sqrt{2}/3$ in (ii) of theorem 3.2, then we get the curve β^* as follows:

$$\beta^*(s) = \beta(s) - \frac{\sqrt{2}}{3}N(s) = \left(\frac{2\sqrt{2}}{3}\sinh s, \frac{2\sqrt{2}}{3}\cosh s, s\right)$$

By straight calculations, we get

$$\begin{aligned} T^*(s) &= \left(2\sqrt{2}\cosh s, 2\sqrt{2}\sinh s, 3\right), \\ N^*(s) &= (\sinh s, \cosh s, 0), \\ B^*(s) &= \left(3\cosh s, 3\sinh s, 2\sqrt{2}\right) \end{aligned}$$

and $\kappa_1^*(s) = 6\sqrt{2}$, $\kappa_2^*(s) = -9$. It can be easily seen that the curve β^* is a spacelike Bertrand mate curve of the curve β.

Example 3. For the same timelike curve β in Example 1, if we take $\lambda = 1 - \sqrt{2}$ in theorem 3.4, then we get the curve β^* as follows:

$$\beta^* (s) = \beta (s) + \left(1 - \sqrt{2}\right) N (s) = (\sinh s, \cosh s, s)$$

By straight calculations, we get

$$T^*(s) = (\cosh s, \sinh s, 1) ,$$
$$N^* (s) = (\sinh s, \cosh s, 0) ,$$
$$B^* (s) = \left(-\frac{\cosh s}{2}, -\frac{\sinh s}{2}, \frac{1}{2}\right)$$

and $\kappa_1^* (s) = 1$, $\kappa_2^* (s) = 1/2$. It can be easily seen that the curve β^* is a Cartan null Bertrand mate curve of the curve β.

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References

[1] H. Balgetir, M. Bektaş and M. Ergüt, *Bertrand curves for non-null curves in 3-dimensional Lorentzian space*, Hadronic J. **27(2)** (2004), 229-236.

[2] H. Balgetir, M. Bektaş and J. Inoguchi, *Null Bertrand curves in Minkowski 3-space and their characterizations*, Note Mat. **23(1)** (2004/05), 7-13.

[3] J. M. Bertrand, *Mémoire sur la théorie des courbes á double courbure*, Comptes Rendus **36** (1850).

[4] W. B. Bonnor,*Null curves in a Minkowski space-time*, Tensor **20** (1969), 229-242.

[5] W. B. Bonnor,*Curves with null normals in Minkowski space-time*, A random walk in relativity and cosmology, Wiley Easten Limited (1985), 33-47.

[6] N. Ekmekci and K. İlarslan,*On Bertrand curves and their characterization*, Differ. Geom. Dyn. Syst. **3(2)** (2001), 17-24.

[7] D. H. Jin, *Null Bertrand curves in a Lorentz manifold*, J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math. **15(3)** (2008), 209-215.

[8] W. Kuhnel, *Differential geometry: curves-surfaces-manifolds*, Braunschweig, Wiesbaden, 1999.

[9] H. Matsuda and S. Yorozu, *Notes on Bertrand curves*, Yokohama Math. J. **50(1-2)** (2003), 41-58.

[10] L. R. Pears,*Bertrand curves in Riemannian space*, J. London Math. Soc. **1-10(2)** (1935), 180-183.

[11] A. Uçum, O. Keçilioğlu and K. İlarslan, *Generalized Bertrand curves with timelike (1,3)-normal plane in Minkowski space-time*, Kuwait J. Sci. **42(3)** (2015), 10-27.

[12] A. Uçum, O. Keçilioğlu and K. İlarslan, *Generalized pseudo null Bertrand curves in semi-Euclidean 4-space with index 2*, Rend. Circ. Mat. Palermo (2016), DOI 10.1007/s12215-016-0246-x.

[13] A. Uçum, K. İlarslan and M. Sakaki, *On* $(1,3)$*-Cartan null Bertrand curves in Semi-Euclidean 4-Space with index 2*, J. Geom. (2015), DOI 10.1007/s00022-015-0290-2.

[14] B. Saint Venant, *Mémoire sur les lignes courbes non planes*, Journal de l'Ecole Polytechnique **18** (1845), 1-76.

[15] C. Xu, X. Cao and P. Zhu,*Bertrand curves and Razzaboni surfaces in Minkowski 3-space*, Bull. Korean Math. Soc. **52(2)** (2015), 377-394.

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