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Fixed points of ordered F-contractions

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Abstract

In his recent paper, Wardowski [16] introduced the concept of F-contraction, which is a proper generalization of ordinary contraction on a complete metric space. Then, some generalizations of F-contractions including multivalued case are obtained in [2, 4, 7, 13]. In this paper, by considering both F-contractions and fixed point result on ordered metric spaces, we introduce a new concept of ordered F-contraction on ordered metric space. Then, we give a fixed point theorem for such mapping. To support our result, we give an example showing that our main theorem is applicable, but both results of Ran and Reurings [12] and Wardowski [16] are not.

Keywords: Fixed point, F-contraction, ordered F-contraction, complete metric space.

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1. Introduction and preliminaries

Recently, combining the ideas of Tarski's fixed point theorem on ordered sets and Banach fixed point theorem on a complete metric space, Ran and Reurings [12] obtained a fixed point result on an ordered complete metric space as follows:

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1.1. Theorem. *Let $(X, \preceq)$ be an ordered set and d be a metric on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping such that there exists $L \in [0, 1)$ with*

$$d(Tx, Ty) \leq Ld(x, y) \text{ for all } x, y \in X \text{ with } x \preceq y. \tag{1.1}$$

If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point.

In this theorem, the usual contraction of Banach fixed point principle is weakened but at the expense that the operator is monotone. Then many fixed point theorists such as Abbas et al. [1], Agarwal et al. [3], Ćirić et al. [5], Kumam et al. [6], Nashine and Altun [8] and O'Regan and Petruşel [11] focused on this interesting result and obtained a lot of generalizations and variants. For example, taking the regularity of the space, which will be define thereinafter, instead of continuity of T, Nieto [9] obtained a parallel result. There are several applications of the theorems in this direction to linear and nonlinear matrix equations, differential equations and integral equations (See for example [10, 12]).

On the other hand, in 2012, one of the most popular fixed point theorem on a complete metric space is given by Wardowski [16]. For this, he introduced the concept of F-contraction, which is a proper generalization of ordinary contraction. For the sake of completeness we recall this concept. Let $\mathcal{F}$ be the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

- $(F1)$ F is strictly increasing, i.e., for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta, F(\alpha) < F(\beta)$,
- $(F2)$ for each sequence $\{a_n\}$ of positive numbers,

$$\lim_{n\to\infty} a_n = 0 \text{ if and only if } \lim_{n\to\infty} F(a_n) = -\infty,$$

- $(F3)$ there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Some examples of the functions belonging $\mathcal{F}$ are $F_1(\alpha) = \ln \alpha$, $F_2(\alpha) = \alpha + \ln \alpha$, $F_3(\alpha) = -\frac{1}{\sqrt{\alpha}}$ and $F_4(\alpha) = \ln\left(\alpha^2 + \alpha\right)$.

1.2. Definition ([16])**.** Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be an F-contraction if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that

$$\forall x, y \in X \ [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y))]. \tag{1.2}$$

If we take $F(\alpha) = \ln \alpha$ in Definition 1.2, the inequality (1.2) turns to

$$d(Tx, Ty) \leq e^{-\tau} d(x, y), \text{ for all } x, y \in X, Tx \neq Ty.$$

It is clear that for $x, y \in X$ such that $Tx = Ty$, the inequality $d(Tx, Ty) \leq e^{-\tau} d(x, y)$ also holds. Therefore T is an ordinary contraction with contractive constant $L = e^{-\tau}$. Therefore every ordinary contraction is also F-contraction, but the converse may not be true as shown in the Example 2.5 of [16]. If we choose $F(\alpha) = \alpha + \ln \alpha$, the inequality (1.2) turns to

$$\frac{d(Tx, Ty)}{d(x, y)} e^{d(Tx,Ty)-d(x,y)} \leq e^{-\tau}, \text{ for all } x, y \in X, Tx \neq Ty.$$

In addition, Wardowski concluded that every F-contraction T is a contractive mapping, i.e.,

$$d(Tx, Ty) < d(x, y), \text{ for all } x, y \in X, Tx \neq Ty.$$

Thus, every F-contraction is a continuous map. Also, Wardowski concluded that if $F_1, F_2 \in \mathcal{F}$ with $F_1(\alpha) \leq F_2(\alpha)$ for all $\alpha > 0$ and $G = F_2 - F_1$ is nondecreasing, then every F_1-contraction T is an F_2-contraction.

The following theorem is main result of Wardowski [16];

1.3. Theorem. *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point in X.*

Considering Theorem 1.3, some extensions and generalizations are obtained in [2, 4, 7, 13, 14, 15]. The aim of this paper is to introduce the concept of ordered F-contractions on ordered metric space, by taking into account the ideas of Wardowski [16] and Ran and Reurings [12].

2. The results

Let $(X, \preceq)$ be an ordered set and d be a metric on X, then we will say that the tripled $(X, \preceq, d)$ is an ordered metric space. If (X, d) is complete, then $(X, \preceq, d)$ will be called ordered complete metric space. We will say that X is regular, if the ordered metric space $(X, \preceq, d)$ provides the following condition:

$$\begin{cases} \text{If } \{x_n\} \subset X \text{ is a non-decreasing sequence with } x_n \to x \text{ in } X, \\ \text{then } x_n \preceq x \text{ for all } n. \end{cases}$$

2.1. Definition. Let $(X, \preceq, d)$ be an ordered metric space and $T : X \to X$ be a mapping. Let

$$Y = \{(x, y) \in X \times X : x \preceq y, d(Tx, Ty) > 0\}.$$

We say that T is an ordered F-contraction if $F \in \mathfrak{F}$ and there exists $\tau > 0$ such that

$$\forall (x, y) \in Y \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)). \tag{2.1}$$

2.2. Theorem. *Let $(X, \preceq, d)$ be an ordered complete metric space and $T : X \to X$ be an ordered F-contraction. Let T is nondecreasing mapping and there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. If T is continuous or X is regular, then T has a fixed point.*

Proof. Let $x_0 \in X$ be as mentioned in the hypotheses. We define a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T and so the proof is completed. Thus, suppose that for every $n \in \mathbb{N}$, $x_{n+1} \neq x_n$. Since $x_0 \preceq Tx_0$ and T is nondecreasing, we obtain

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots.$$

Now, since $x_n \preceq x_{n+1}$ and $d(Tx_n, Tx_{n-1}) > 0$ for every $n \in \mathbb{N}$, then $(x_n, x_{n+1}) \in Y$, and so, we can use the inequality (2.1) for the consecutive terms of $\{x_n\}$, then we have

$$F(d(x_{n+1}, x_n)) = F(d(Tx_n, Tx_{n-1})) \leq F(d(x_n, x_{n-1})) - \tau. \tag{2.2}$$

Denote $\gamma_n = d(x_n, x_{n+1})$ for $n \in \mathbb{N}$. Then, from (2.2) we obtain

$$F(\gamma_n) \leq F(\gamma_{n-1}) - \tau \leq F(\gamma_{n-2}) - 2\tau \leq \cdots \leq F(\gamma_0) - n\tau. \tag{2.3}$$

From (2.3), we get $\lim_{n\to\infty} F(\gamma_n) = -\infty$. Thus, from (F2), we have

$$\lim_{n\to\infty} \gamma_n = 0.$$

From (F3), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} \gamma_n^k F(\gamma_n) = 0.$$

By (2.2), the following holds for all $n \in \mathbb{N}$

$$\gamma_n^k F(\gamma_n) - \gamma_n^k F(\gamma_0) \leq -\gamma_n^k n\tau \leq 0. \tag{2.4}$$

Letting $n \to \infty$ in (2.4), we obtain that

$$\lim_{n\to\infty} n\gamma_n^k = 0. \tag{2.5}$$

From (2.5), there exits $n_1 \in \mathbb{N}$ such that $n\gamma_n^k \leq 1$ for all $n \geq n_1$. So, we have

$$\gamma_n \leq \frac{1}{n^{1/k}}, \tag{2.6}$$

for all $n \geq n_1$. In order to show that $\{x_n\}$ is a Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the metric and from (2.6), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \gamma_n + \gamma_{n+1} + \cdots + \gamma_{m-1} \\ &= \sum_{i=n}^{m-1} \gamma_i \\ &\leq \sum_{i=n}^{\infty} \gamma_i \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

Passing to limit $n \to \infty$, we get $d(x_n, x_m) \to 0$. This yields that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, the sequence $\{x_n\}$ converges to some point $z \in X$.

Now, if T is continuous, then we have

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T \lim_{n\to\infty} x_n = Tz$$

and so z is a fixed point of T.

Now suppose X is regular, then $x_n \preceq z$ for all $n \in \mathbb{N}$. We will consider the following two cases:

Case 1. If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0} = z$, then we obtain

$$Tz = Tx_{n_0} = x_{n_0+1} \preceq z.$$

Also, since $x_{n_0} \preceq x_{n_0+1}$, then $z \preceq Tz$ and thus, $z = Tz$.

Case 2. Now, suppose that $x_n \neq z$ for every $n \in \mathbb{N}$ and $d(z, Tz) > 0$. Since $\lim_{n\to\infty} x_n = z$, then there exists $n_1 \in \mathbb{N}$ such that $d(x_{n+1}, Tz) > 0$ and $d(x_n, z) < \frac{d(z,Tz)}{2}$ for all $n \geq n_1$. Note that in this case $(x_n, z) \in Y$. Therefore, by considering (F1), we have, for $n \geq n_1$,

$$\tau + F(d(Tx_n, Tz)) \leq F(d(x_n, z)) \leq F(\frac{d(z,Tz)}{2}),$$

which yields

$$d(x_{n+1}, Tz) \leq \frac{d(z,Tz)}{2}. \tag{2.7}$$

Taking limit as $n \to \infty$, we deduce that

$$d(z, Tz) \leq \frac{d(z,Tz)}{2},$$

a contradiction. Therefore, we conclude that $d(z, Tz) = 0$, i.e. $z = Tz$. □

2.3. Example. Let $A = \{\frac{1}{n^2} : n \in \mathbb{N}\} \cup \{0\}$, $B = \{2, 3\}$ and $X = A \cup B$. Define an order relation $\preceq$ on X as

$$x \preceq y \Leftrightarrow [x = y \text{ or } x, y \in A \text{ with } x \leq y],$$

where $\leq$ is usual order. Obviously, $(X, \preceq, d)$ is ordered complete metric space with the usual metric d. Let $T : X \to X$ be given by

$$Tx = \begin{cases} \frac{1}{(n+1)^2} & , \quad x = \frac{1}{n^2} \\ x & , \quad x \in \{0, 2, 3\} \end{cases}.$$

It is easy to see that T is nondecreasing. Also, for $x_0 = 0$ we have $x_0 \preceq Tx_0$.

On the other side, taking F with

$$F(\alpha) = \begin{cases} \frac{\ln \alpha}{\sqrt{\alpha}} & , \quad 0 < \alpha < e^2 \\ \alpha - e^2 + \frac{2}{e} & , \quad \alpha \geq e^2 \end{cases}.$$

It is easy to see that the conditions $(F1), (F2)$ and $(F3)$ (for $k = \frac{2}{3}$) are satisfied. We claim that T is an ordered F-contraction with $\tau = \ln 2$. To see this, let us consider the following calculations:

It is obvious that

$$\begin{aligned} Y &= \{(x, y) \in X \times X : x \preceq y, d(Tx, Ty) > 0\} \\ &= \{(x, y) \in X \times X : x, y \in A \text{ and } x < y\}. \end{aligned}$$

Therefore, to see (2.1), it is sufficient to show that

$$\forall (x, y) \in Y \Rightarrow \ln 2 + F(d(Tx, Ty)) \leq F(d(x, y))$$

$$\Leftrightarrow x, y \in A \text{ and } x < y \Rightarrow d(Tx, Ty)^{\frac{1}{\sqrt{d(Tx,Ty)}}} d(x, y)^{-\frac{1}{\sqrt{d(x,y)}}} \leq \frac{1}{2} \tag{2.8}$$

$$\Leftrightarrow x, y \in A \text{ and } x < y \Rightarrow |Tx - Ty|^{\frac{1}{\sqrt{|Tx-Ty|}}} |x - y|^{-\frac{1}{\sqrt{|x-y|}}} \leq \frac{1}{2}.$$

Using Example 5 of [7], we can see that (2.8) is true. Also, T is continuous (and X is regular). Therefore, all conditions of Theorem 2.2 are satisfied, and so, T has a fixed point in X.

On the other hand, since $0 \preceq \frac{1}{n^2}$ for all $n \in \mathbb{N}$ and

$$\sup_{n \in \mathbb{N}} \frac{d(T0, T\frac{1}{n^2})}{d(0, \frac{1}{n^2})} = \sup_{n \in \mathbb{N}} \frac{n^2}{(n+1)^2} = 1,$$

then Theorem 1.1 , which is main result of [12], is not applicable to this example. Again, since

$$d(T2, T3) = d(2, 3) = 1,$$

then for all $F \in \mathcal{F}$ and $\tau > 0$ we have

$$\tau + F(d(T2, T3)) > F(d(2, 3)).$$

Therefore, Theorem 1.3, which is main result of [16], is not applicable to this example.

2.4. Remark. In Theorem 2.2, if we assume the following condition:

(2.9) every pair of elements has a lower bound and upper bound,

then, the fixed point of T is unique. To see this, it is sufficient to show that for every $x \in X$,

$$\lim_{n \to \infty} T^n x = z,$$

where z is the fixed point of T such that

$$z = \lim_{n \to \infty} T^n x_0.$$

For this we will consider the following cases: Let $x \in X$ and x_0 be as in Theorem 2.2.

Case 1. If $x \preceq x_0$ or $x_0 \preceq x$, then $T^n x \preceq T^n x_0$ or $T^n x_0 \preceq T^n x$ for all $n \in \mathbb{N}$. If $T^{n_0} x = T^{n_0} x_0$ for some $n_0 \in \mathbb{N}$, then $T^n x \to z$. Now let $T^n x_0 \neq T^n x$ for all $n \in \mathbb{N}$, then $d(T^n x_0, T^n x) > 0$ and so $(T^n x_0, T^n x) \in Y$ for all $n \in \mathbb{N}$. Therefore from (2.1), we have

$$\begin{aligned} F(d(T^n x_0, T^n x)) &\leq F(d(T^{n-1} x_0, T^{n-1} x)) - \tau \\ &\leq F(d(T^{n-2} x_0, T^{n-2} x)) - 2\tau \\ &\;\;\vdots \\ &\leq F(d(x_0, x)) - n\tau. \end{aligned} \tag{2.10}$$

Taking into account (F2), from (2.10) we have $\lim_{n\to\infty} d(T^n x_0, T^n x) = 0$, and so, $\lim_{n\to\infty} T^n x_0 = \lim_{n\to\infty} T^n x = z$.

Case 2. If $x \npreceq x_0$ or $x_0 \npreceq x$, then from (2.9), there exist $x_1, x_2 \in X$ such that

$$x_2 \preceq x \preceq x_1 \text{ and } x_2 \preceq x_0 \preceq x_1.$$

Therefore, as in the Case 1, we can show that

$$\lim_{n\to\infty} T^n x_1 = \lim_{n\to\infty} T^n x_2 = \lim_{n\to\infty} T^n x = \lim_{n\to\infty} T^n x_0 = z.$$

2.5. Remark. As we can see in the Example 2.3, if the condition (2.9) is not satisfied, then the fixed point of T may not be unique.

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