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SOME APPROXIMATION PROPERTIES OF KANTOROVICH VARIANT OF CHLODOWSKY OPERATORS BASED ON $q-$INTEGER

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Abstract. In this paper, we introduce two different Kantorovich type generalization of the $q-$Chlodowsky operators. For the first operators we give some weighted approximation theorems and a Voronovskaja type theorem. Also, we present the local approximation properties and the order of convergence for unbounded functions of these operators . For second operators, we obtain a weighted statistical approximation property.

1. Introduction

In 1997, G. Phillips [21] introduced the generalization of Bernstein polynomials based on $q-$integers. The author estimated the rate of convergence and obtained a Voronovskaja-type theorem for the generalization of Bernstein operators. Recently, generalizations of positive linear operators based on $q-$integers were defined and studied by several authors. For example; Karsli and Gupta [3] introduced the following $q-$Chlodowsky polynomials defined as:

$$\left(C_{n,q}f\right)(x)=\sum_{k=0}^{n}f\left(\frac{[k]_q}{[n]_q}b_n\right)\left[\begin{array}{c}n\\k\end{array}\right]_q\left(\frac{x}{b_n}\right)^k\prod_{i=0}^{n-k-1}\left(1-q^i\frac{x}{b_n}\right)^{n-k},\ 0\leqslant x\leqslant b_n,$$

where b_n is a positive increasing sequence with $b_n\to\infty$. They investigated the rate of convergence and the monotonicity property of these operators. For more works, see references [4, 5, 6, 7, 8, 9, 10].

In this study, we define Kantorovich type generalization of $q-$Chlodowsky operators. We examine the statical approximation properties of our new operator by the help of Korovkin-type theorem in weighted space. Further, we present the local approximation properties and the order of convergence for unbounded functions of

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these operators. Furthermore, we prove and state a Voronovskaja type theorem for our new operators.

The Kantorovich type generalization of $q-$ Chlodowsky operators as follows:

$$C_n^q(f;x) = [n]_q \sum_{k=0}^{n} q^{-k} \begin{bmatrix} n \\ k \end{bmatrix}_q P_{n,k}^q(x) \int\limits_{\frac{[k]_q}{[n]_q}}^{\frac{[k+1]_q}{[n]_q}} f(b_n t) d_q t, \tag{1.1}$$

where $n \geqslant 1$, $q \in (0,1]$ and $P_{n,k}^q(x) = \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)_q^{n-k}$.

Assume that f is a monotone increasing function on $[0, b_n]$, then using (1.2) one can easily verified that $C_n^q(f;x)$ are linear and positive operators for $0 < q \leqslant 1$.

Let us recall some definitions and notations regarding the concept of $q-$ calculus. For any fixed real number $q > 0$ and non-negative integer, the $q-$integer of the number n is defined by

$$[n]_q := \begin{cases} (1-q^n)/(1-q), & q \neq 1 \\ 1, & q = 0 \end{cases}.$$

The $q-$factorial $[n]_q!$ is defined as following

$$[n]_q! := \begin{cases} [n]_q [n-1]_q \cdots [1]_q, & n \in \mathbb{N} \\ 1, & n = 0 \end{cases}.$$

The $q-$binomial coefficients are also defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}, 0 \leqslant k \leqslant n.$$

The $q-$analogue of the integration in the interval $[0, b]$ is defined as (see [11])

$$\int\limits_0^b f(t) d_q t = (1-q) b \sum_{j=0}^{\infty} f(q^j b) q^j, \ 0 < q < 1.$$

Over a general interval $[a, b]$, one can write

$$\int\limits_a^b f(t) d_q t = \int\limits_0^b f(t) d_q t - \int\limits_0^a f(t) d_q t. \tag{1.2}$$

Further results related to $q-$calculus can be found in [1, 2].

2. Some Basic Results

We need the following lemmas for proving our main results.

Lemma 2.1. *By the definition of $q-$integral, we have*

$$\int_{\frac{[k]_q}{[n]_q}}^{\frac{[k+1]_q}{[n]_q}} d_qt = \frac{q^k}{[n]_q},$$

$$\int_{\frac{[k]_q}{[n]_q}}^{\frac{[k+1]_q}{[n]_q}} b_n t d_qt = \frac{b_n q^k}{[n]_q^2[2]_q}\left([2]_q[k]_q+1\right),$$

$$\int_{\frac{[k]_q}{[n]_q}}^{\frac{[k+1]_q}{[n]_q}} b_n^2 t^2 d_qt = \frac{b_n^2 q^k}{[n]_q^3[3]_q}\Big([3]_q[k]_q^2+(2q+1)[k]_q+1\Big),$$

$$\int_{\frac{[k]_q}{[n]_q}}^{\frac{[k+1]_q}{[n]_q}} b_n^3 t^3 d_qt = \frac{b_n^3 q^k}{[n]_q^4[4]_q}\Big([4]_q[k]_q^3+\big(3q^2+2q+1\big)[k]_q^2+(3q+1)[k]_q+1\Big),$$

$$\int_{\frac{[k]_q}{[n]_q}}^{\frac{[k+1]_q}{[n]_q}} b_n^4 t^4 d_qt = \frac{b_n^4 q^k}{[n]_q^5[5]_q}\Big\{[5]_q[k]_q^4+\big(4q^3+3q^2+2q+1\big)[k]_q^3$$
$$+\big(6q^2+3q+1\big)[k]_q^2+(4q+1)[k]_q+1\Big\}.$$

Lemma 2.2. *The following equalities hold.*

$$\sum_{k=0}^{n} b_n\frac{[k]_q}{[n]_q}\left[\begin{array}{c} n \\ k \end{array}\right]_q P_{n,k}^q(x) = x,$$

$$\sum_{k=0}^{n} b_n^2\frac{[k]_q^2}{[n]_q^2}\left[\begin{array}{c} n \\ k \end{array}\right]_q P_{n,k}^q(x) = \frac{q[n-1]_q}{[n]_q}x^2+\frac{b_n}{[n]_q}x,$$

$$\sum_{k=0}^{n} b_n^3\frac{[k]_q^3}{[n]_q^3}\left[\begin{array}{c} n \\ k \end{array}\right]_q P_{n,k}^q(x) = \frac{q^3[n-1]_q[n-2]_q}{[n]_q^2}x^3+\frac{\big(q^2+2q\big)[n-1]_q b_n}{[n]_q^2}x^2$$
$$+\frac{b_n^2}{[n]_q^2}x,$$

$$\sum_{k=0}^{n} b_n^4 \frac{[k]_q^4}{[n]_q^4} \left[\begin{array}{c} n \\ k \end{array}\right]_q P_{n,k}^q(x) = \frac{q^6 [n-1]_q [n-2]_q [n-3]_q}{[n]_q^3} x^4 + \frac{q^3 \left(q^2+2q+3\right) [n-1]_q [n-2]_q b_n}{[n]_q^3} x^3 + \frac{q\left(q^2+3q+3\right)[n-1]_q b_n^2}{[n]_q^3} x^2 + \frac{b_n^3}{[n]_q^3} x.$$

Proof. Using the equality

$$[n]_q = [j]_q + q^j [n-j]_q, \;\; 0 \leqslant j \leqslant n, \tag{2.1}$$

we have

$$\begin{aligned}
\sum_{k=0}^{n} b_n \frac{[k]_q}{[n]_q} \left[\begin{array}{c} n \\ k \end{array}\right]_q P_{n,k}^q(x) &= x, \\
\sum_{k=0}^{n} b_n^2 \frac{[k]_q^2}{[n]_q^2} \left[\begin{array}{c} n \\ k \end{array}\right]_q P_{n,k}^q(x) &= \sum_{k=1}^{n} b_n^2 \frac{[k]_q}{[n]_q} \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_q P_{n,k}^q(x) \\
&= \sum_{k=1}^{n} b_n^2 \frac{q[k-1]_q+1}{[n]_q} \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_q P_{n,k}^q(x) \\
&= \frac{qb_n^2}{[n]_q} \sum_{k=1}^{n} [k-1]_q \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_q P_{n,k}^q(x) + \frac{b_n}{[n]_q} x \\
&= \frac{qb_n^2 [n-1]_q}{[n]_q} \sum_{k=2}^{n} \left[\begin{array}{c} n-2 \\ k-2 \end{array}\right]_q P_{n,k}^q(x) + \frac{b_n}{[n]_q} x \\
&= \frac{qb_n^2 [n-1]_q}{[n]_q} \sum_{k=0}^{n} \left[\begin{array}{c} n-2 \\ k \end{array}\right]_q P_{n,k+2}^q(x) + \frac{b_n}{[n]_q} x \\
&= \frac{q[n-1]_q}{[n]_q} x^2 + \frac{b_n}{[n]_q} x
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{n} b_n^3 \frac{[k]_q^3}{[n]_q^3} \left[\begin{array}{c} n \\ k \end{array}\right]_q P_{n,k}^q(x) &= \sum_{k=1}^{n} b_n^3 \frac{[k]_q^2}{[n]_q^2} \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_q P_{n,k}^q(x) \\
&= \frac{b_n^3 q^2}{[n]_q^2} \sum_{k=1}^{n} [k-1]_q^2 \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_q P_{n,k}^q(x)
\end{aligned}$$

$$+\frac{b_n^3 2q}{[n]_q^2} \sum_{k=1}^{n} [k-1]_q \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_q P_{n,k}^q(x) + \frac{b_n^3}{[n]_q^2} \sum_{k=1}^{n} \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_q P_{n,k}^q(x)$$

$$\begin{aligned}
&= \frac{q^2 b_n^3 [n-1]_q}{[n]_q^2} \sum_{k=2}^{n} [k-1]_q \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q P_{n,k}^q(x) \\
&\quad + \frac{2q b_n^3 [n-1]_q}{[n]_q^2} \sum_{k=2}^{n} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q P_{n,k}^q(x) + \frac{b_n^2}{[n]_q^2} x \\
&= \frac{q^3 b_n^3 [n-1]_q}{[n]_q^2} \sum_{k=2}^{n} [k-2]_q \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q P_{n,k}^q(x) \\
&\quad + \frac{q^2 b_n^3 [n-1]_q}{[n]_q^2} \sum_{k=2}^{n} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q P_{n,k}^q(x) + \frac{2q b_n [n-1]_q}{[n]_q^2} x^2 + \frac{b_n^2}{[n]_q^2} x \\
&= \frac{q^3 [n-1]_q [n-2]_q}{[n]_q^2} x^3 + \frac{\left(q^2+2q\right) b_n [n-1]_q}{[n]_q^2} x^2 + \frac{b_n^2}{[n]_q^2} x.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\sum_{k=0}^{n} b_n^4 \frac{[k]_q^4}{[n]_q^4} \begin{bmatrix} n \\ k \end{bmatrix}_q P_{n,k}^q(x) &= \sum_{k=1}^{n} b_n^4 \frac{[k]_q^3}{[n]_q^3} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q P_{n,k}^q(x) \\
&= \frac{q^3 b_n^4}{[n]_q^3} \sum_{k=1}^{n} [k-1]_q^3 \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q P_{n,k}^q(x) \\
&\quad + \frac{3q^2 b_n^4}{[n]_q^3} \sum_{k=1}^{n} [k-1]_q^2 \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q P_{n,k}^q(x) \\
&\quad + \frac{3q b_n^4}{[n]_q^3} \sum_{k=1}^{n} [k-1]_q \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q P_{n,k}^q(x) + \frac{b_n^3}{[n]_q^3} x \\
&= \frac{q^3 b_n^4 [n-1]_q}{[n]_q^3} \sum_{k=2}^{n} [k-1]_q^2 \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q P_{n,k}^q(x) \\
&\quad + \frac{3q^2 b_n^4 [n-1]_q}{[n]_q^3} \sum_{k=2}^{n} [k-1]_q \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q P_{n,k}^q(x) \\
&\quad + \frac{3q b_n^2 [n-1]_q}{[n]_q^3} x^2 + \frac{b_n^3}{[n]_q^3} x \\
&= \frac{q^5 b_n^4 [n-1]_q}{[n]_q^3} \sum_{k=2}^{n} [k-2]_q^2 \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q P_{n,k}^q(x) \\
&\quad + \frac{2q^4 b_n^4 [n-1]_q}{[n]_q^3} \sum_{k=2}^{n} [k-2]_q \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q P_{n,k}^q(x)
\end{aligned}$$

$$
\begin{aligned}
& +\frac{q^3 b_n^2 [n-1]_q}{[n]_q^3} x^2 + \frac{3q^3 [n-1]_q [n-2]_q b_n}{[n]_q^3} x^3 \\
& +\frac{3q^2 [n-1]_q b_n^2}{[n]_q^3} x^2 + \frac{3q [n-1]_q b_n^2}{[n]_q^3} x^2 + \frac{b_n^3}{[n]_q^3} x \\
= & \frac{q^5 [n-1]_q [n-2]_q b_n^4}{[n]_q^3} \sum_{k=3}^{n} [k-2]_q \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}_q P_{n,k}^q(x) \\
& +\frac{\left(2q^4+3q^3\right) b_n [n-1]_q [n-2]_q}{[n]_q^3} x^3 + \frac{\left(3q^2+3q\right) b_n^2 [n-1]_q}{[n]_q^3} x^2 + \frac{b_n^3}{[n]_q^3} x \\
= & \frac{q^6 [n-1]_q [n-2]_q [n-3]_q}{[n]_q^3} x^4 + \frac{q^3\left(q^2+2q+3\right) [n-1]_q [n-2]_q b_n}{[n]_q^3} x^3 \\
& +\frac{q\left(q^2+3q+3\right) [n-1]_q b_n^2}{[n]_q^3} x^2 + \frac{b_n^3}{[n]_q^3} x.
\end{aligned}
$$

□

Lemma 2.3. *The operators defined by (1.1) satisfy the following properties:*

$$
\begin{aligned}
C_n^q(1;x) &= 1, \\
C_n^q(t;x) &= x + \frac{K_0 b_n}{[n]_q}, \qquad (2.2) \\
C_n^q\left(t^2;x\right) &= \frac{q[n-1]_q}{[n]_q} x^2 + \frac{K_1 b_n}{[n]_q} x + \frac{K_2 b_n^2}{[n]_q^2}, \qquad (2.3) \\
C_n^q\left(t^3;x\right) &= \frac{q^3 [n-1]_q [n-2]_q}{[n]_q^2} x^3 + \frac{qK_3 [n-1]_q b_n}{[n]_q^2} x^2 + \frac{K_4 b_n^2}{[n]_q^2} x + \frac{b_n^3}{[4]_q [n]_q^3}, \\
C_n^q\left(t^4;x\right) &= \frac{q^6 [n-1]_q [n-2]_q [n-2]_q}{[n]_q^3} x^4 + \frac{q^3 K_5 [n-1]_q [n-2]_q b_n}{[n]_q^3} x^3 \\
& \quad + \frac{qK_6 [n-1]_q b_n^2}{[n]_q^3} x^2 + \frac{K_7 b_n^3}{[n]_q^3} x + \frac{b_n^4}{[5]_q [n]_q^4},
\end{aligned}
$$

for all $x \in [0, b_n]$*, where*

$$
\begin{aligned}
K_0 &= \frac{1}{[2]_q},\ K_1 = \frac{q^2+3q+2}{[3]_q},\ K_2 = \frac{1}{[3]_q}, \\
K_4 &= \frac{q^3+4q^2+6q+2}{[4]_q},\ K_3 = \frac{q^4+3q^3+6q^2+5q+3}{[4]_q}, \\
K_5 &= \frac{q^6+3q^5+6q^4+10q^3+9q^2+7q+4}{[5]_q},
\end{aligned}
$$

$$
\begin{aligned}
K_6 &= \frac{q^6+4q^5+11q^4+18q^3+15q^2+11q+5}{[5]_q}, \\
K_7 &= \frac{q^4+5q^3+10q^2+10q+4}{[5]_q}.
\end{aligned}
$$

Proof. By using definition of $C_n^q(f,x)$, Lemma 2.1 and Lemma 2.2, we get

$$
\begin{aligned}
C_n^q(1;x) &= [n]_q \sum_{k=0}^{n} q^{-k} \left[\begin{array}{c} n \\ k \end{array} \right]_q P_{n,k}^q(x) \frac{q^k}{[n]_q} = 1, \\
C_n^q(t;x) &= [n]_q \sum_{k=0}^{n} q^{-k} \left[\begin{array}{c} n \\ k \end{array} \right]_q P_{n,k}^q(x) \frac{q^k b_n}{[2]_q [n]_q^2} \left([2]_q [k]_q + 1 \right) \\
&= \sum_{k=0}^{n} b_n \frac{[k]_q}{[n]_q} \left[\begin{array}{c} n \\ k \end{array} \right]_q P_{n,k}^q(x) + \frac{b_n}{[2]_q [n]_q} \\
&= x + \frac{K_0 b_n}{[n]_q}, \\
C_n^q(t^2;x) &= [n]_q \sum_{k=0}^{n} q^{-k} \left[\begin{array}{c} n \\ k \end{array} \right]_q P_{n,k}^q(x) \frac{q^k b_n^2}{[3]_q [n]_q^3} \left([3]_q [k]_q^2 + (2q+1)[k]_q + 1 \right) \\
&= b_n^2 \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q P_{n,k}^q(x) + \frac{(2q+1) b_n^2}{[3]_q [n]_q^3} \sum_{k=0}^{n} \frac{[k]_q}{[n]_q} \left[\begin{array}{c} n \\ k \end{array} \right]_q P_{n,k}^q(x) + \frac{b_n^2}{[3]_q [n]_q^2} \\
&= \frac{q[n-1]_q}{[n]_q} x^2 + \frac{K_1 b_n}{[n]_q} x + \frac{K_2 b_n^2}{[n]_q^2}.
\end{aligned}
$$

For t^3, we get

$$
\begin{aligned}
C_n^q(t^3;x) &= [n]_q \sum_{k=0}^{n} q^{-k} \left[\begin{array}{c} n \\ k \end{array} \right]_q P_{n,k}^q(x) \frac{q^k b_n^3}{[4]_q [n]_q^4} \Big\{ [4]_q [k]_q^3 + \left(3q^2+2q+1\right)[k]_q^2 \\
&\quad + (3q+1)[k]_q + 1 \Big\} \\
&= b_n^3 \sum_{k=0}^{n} \frac{[k]_q^3}{[n]_q^3} \left[\begin{array}{c} n \\ k \end{array} \right]_q P_{n,k}^q(x) + \frac{\left(3q^2+2q+1\right) b_n^3}{[4]_q [n]_q^4} \sum_{k=0}^{n} \frac{[k]_q^2}{[n]_q^2} \left[\begin{array}{c} n \\ k \end{array} \right]_q P_{n,k}^q(x) \\
&\quad + \frac{(3q+1) b_n^3}{[4]_q [n]_q^2} \sum_{k=0}^{n} \frac{[k]_q}{[n]_q} \left[\begin{array}{c} n \\ k \end{array} \right]_q P_{n,k}^q(x) + \frac{b_n^3}{[4]_q [n]_q^3} \\
&= \frac{q^3 [n-1]_q [n-2]_q}{[n]_q^2} x^3 + \frac{q K_3 [n-1]_q b_n}{[n]_q^2} x^2 + \frac{K_4 b_n^2}{[n]_q^2} x + \frac{b_n^3}{[4]_q [n]_q^3}
\end{aligned}
$$

and finally

$$\begin{aligned}
C_n^q\left(t^4;x\right) &= [n]_q\sum_{k=0}^{n} q^{-k}\left[\begin{array}{c} n \\ k \end{array}\right]_q P_{n,k}^q(x)\,\frac{q^k b_n^4}{[5]_q\,[n]_q^5}\big([5]_q\,[k]_q^4 \\
&\quad+\left(4q^3+3q^2+2q+1\right)[k]_q^3 \\
&\quad+\left(6q^2+3q+1\right)[k]_q^2+(4q+1)\,[k]_q+1\big) \\
&= b_n^4\sum_{k=0}^{n}\frac{[k]_q^4}{[n]_q^4}\left[\begin{array}{c} n \\ k \end{array}\right]_q P_{n,k}^q(x) \\
&\quad+\frac{b_n^4\left(4q^3+3q^2+2q+1\right)}{[5]_q\,[n]_q^4}\sum_{k=0}^{n}\frac{[k]_q^3}{[n]_q^3}\left[\begin{array}{c} n \\ k \end{array}\right]_q P_{n,k}^q(x) \\
&\quad+\frac{b_n^4\left(6q^2+3q+1\right)}{[5]_q\,[n]_q^2}\sum_{k=0}^{n}\frac{[k]_q^2}{[n]_q^2}\left[\begin{array}{c} n \\ k \end{array}\right]_q P_{n,k}^q(x) \\
&\quad+\frac{b_n^4\,(4q+1)}{[5]_q\,[n]_q^3}\sum_{k=0}^{n}\frac{[k]_q}{[n]_q}\left[\begin{array}{c} n \\ k \end{array}\right]_q P_{n,k}^q(x)+\frac{b_n^4}{[5]_q\,[n]_q^4} \\
&= \frac{q^6\,[n-1]_q\,[n-2]_q\,[n-2]_q}{[n]_q^3}x^4+\frac{q^3K_5\,[n-1]_q\,[n-2]_q\,b_n}{[n]_q^3}x^3 \\
&\quad+\frac{qK_6\,[n-1]_q\,b_n^2}{[n]_q^3}x^2+\frac{K_7b_n^3}{[n]_q^3}x+\frac{b_n^4}{[5]_q\,[n]_q^4}.
\end{aligned}$$

□

3. Weighted approximation

We consider the following class of functions:

Let $H_{x^2}[0,\ \infty)$ be the set of all functions f defined on $[0,\ \infty)$ satisfying the condition $|f(x)|\leqslant M_f\left(1+x^2\right)$, where M_f is a constant depending only on f. By $C_{x^2}[0,\ \infty)$, we denote the subspace of all continuous functions belonging to $H_{x^2}[0,\ \infty)$. Also, let $C_{x^2}^*[0,\ \infty)$ be the subspace of all functions $f\in C_{x^2}[0,\ \infty)$, for which $\lim\limits_{x\to\infty}\frac{f(x)}{1+x^2}$ is finite. The norm on $C_{x^2}^*[0,\ \infty)$ is $\|f\|_{x^2}=\sup_{x\in[0,\ \infty)}\frac{|f(x)|}{1+x^2}$.

Now, we shall discuss the weighted approximation theorem, where the approximation formula holds true on the interval $[0,\ \infty)$.

Theorem 3.1. *Let $q=q_n$ satisfy $0<q_n\leqslant 1$ and for n sufficiently large $q_n\to 1$ and $\frac{b_n}{[n]_{q_n}}\to 0$ with $b_n\to\infty$. Let $f\in C_{x^2}^*[0,\ \infty)$ and f be a monotone increasing function on $[0,\ \infty)$. Then we have*

$$\lim_{n\to\infty}\sup_{0\leqslant x\leqslant b_n}\frac{\left|C_n^{q_n}(f;x)-f(x)\right|}{1+x^2}=0$$

Proof. Setting the operators

$$^{*}C_n^{q_n}(f;x)=\begin{cases} C_n^{q_n}(f;x) & \text{if } 0\leqslant x\leqslant b_n \\ f(x) & \text{if } x>b_n \end{cases}$$

and using the theorem in [15] for the operators $^{*}C_n^{q_n}$, we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty}\left\|^{*}C_n^{q_n}(t^{\nu},x)-x^{\nu}\right\|_{x^2}=\lim_{n\to\infty}\sup_{0\leqslant x\leqslant b_n}\frac{\left|C_n^{q_n}(t^{\nu};x)-x^{\nu}\right|}{1+x^2}\qquad \nu=0,1,2. \tag{3.1}$$

Since $C_n^{q_n}(1,x)=1$ the first condition of (3.1) is fulfilled for $\nu=0$.

Using Lemma 2.3, we can write

$$\sup_{0\leqslant x\leqslant b_n}\frac{\left|C_n^{q_n}(t;x)-x\right|}{1+x^2}=\frac{1}{1+q_n}\frac{b_n}{[n]_{q_n}}$$

and

$$\sup_{0\leqslant x\leqslant b_n}\frac{\left|C_n^{q_n}\left(t^2;x\right)-x^2\right|}{1+x^2}\leqslant\sup_{0\leqslant x\leqslant b_n}\frac{x^2}{1+x^2}\frac{b_n}{[n]_{q_n}}+\sup_{0\leqslant x\leqslant b_n}\frac{x}{1+x^2}\frac{q_n^2+3q_n+2}{q_n^2+q_n+1}\frac{b_n}{[n]_{q_n}}$$
$$+\sup_{0\leqslant x\leqslant b_n}\frac{1}{1+x^2}\frac{1}{q_n^2+q_n+1}\frac{b_n^2}{[n]_{q_n}^2}.$$

which implies that

$$\lim_{n\to\infty}\sup_{0\leqslant x\leqslant b_n}\frac{\left|C_n^{q_n}(t;x)-x\right|}{1+x^2}=0$$

and

$$\lim_{n\to\infty}\sup_{0\leqslant x\leqslant b_n}\frac{\left|C_n^{q_n}\left(t^2;x\right)-x^2\right|}{1+x^2}=0.$$

Thus the proof is completed. □

We know that for $f\in C_{x^2}[0,\ \infty)$ Theorem 3.1 is not true (see [15]). But we can give following property of C_n^q.

Theorem 3.2. *Let $q=q_n$ satisfy $0<q_n\leqslant 1$ and for n sufficiently large $q_n\to 1$ and $\frac{b_n}{[n]_{q_n}}\to 0$ with $b_n\to\infty$. Let $f\in C_{x^2}[0,\ \infty)$ and f be a monotone increasing function on $[0,\ \infty)$. Then we have*

$$\lim_{n\to\infty}\frac{1}{\sqrt{b_n}}\sup_{0\leqslant x\leqslant b_n}\frac{\left|C_n^{q_n}(f;x)-f(x)\right|}{1+x^2}=0$$

Proof. f is continuous function we can write

$$|f(t)-f(x)|<\varepsilon$$

if $|t-x|<\delta$ and $|t-x|\geqslant\delta$ we have

$$|f(t)-f(x)|<C_f(\delta)\left((t-x)^2+\left(1+x^2\right)|t-x|\right).$$

Thus we can write

$$|f(t)-f(x)|<\varepsilon+C_f(\delta)\left((t-x)^2+\left(1+x^2\right)|t-x|\right).$$

for $x\in[0,b_n]$ and $t\in[0,\ \infty)$. Since $C_n^{q_n}$ linear and positive operator we have

$$\begin{aligned}\left|C_n^{q_n}(f;x)-f(x)\right| &\leqslant \varepsilon+C_f(\delta)\,C_n^{q_n}\left((t-x)^2;x\right)\\ &\quad +C_f(\delta)\left(1+x^2\right)C_n^{q_n}\left(|t-x|;x\right)\\ &\leqslant \varepsilon+C_f(\delta)\,C_n^{q_n}\left((t-x)^2;x\right)\\ &\quad +C_f(\delta)\left(1+x^2\right)\sqrt{C_n^{q_n}\left((t-x)^2;x\right)}.\end{aligned}$$

From Lemma 2.3, we have

$$\begin{aligned}\frac{\left|C_n^{q_n}(f;x)-f(x)\right|}{1+x^2} &\leqslant \frac{\varepsilon}{1+x^2}+C_f(\delta)\left[\frac{x(3b_n-x)}{(1+x^2)[n]_{q_n}}+\frac{b_n^2}{[n]_{q_n}^2}\right.\\ &\quad\left.+\sqrt{\frac{x(3b_n-x)}{[n]_{q_n}}+\frac{b_n^2}{[n]_{q_n}^2}}\right]\end{aligned}$$

and

$$\begin{aligned}\sup_{0\leqslant x\leqslant b_n}\frac{\left|C_n^{q_n}(f;x)-f(x)\right|}{1+x^2} &\leqslant \varepsilon\sup_{0\leqslant x\leqslant b_n}\frac{1}{1+x^2}+C_f(\delta)\left[\frac{x(3b_n-x)}{(1+x^2)[n]_{q_n}}+\frac{b_n^2}{[n]_{q_n}^2}\right.\\ &\quad\left.+\sqrt{\frac{x(3b_n-x)}{[n]_{q_n}}+\frac{b_n^2}{[n]_{q_n}^2}}\right]\\ &\leqslant \varepsilon+C_f(\delta)\left[\frac{3b_n}{[n]_{q_n}}+\frac{b_n^2}{[n]_{q_n}^2}+\sqrt{\frac{3b_n^2}{[n]_{q_n}}+\frac{b_n^2}{[n]_{q_n}^2}}\right].\end{aligned}$$

Therefore

$$\frac{1}{\sqrt{b_n}}\sup_{0\leqslant x\leqslant b_n}\frac{\left|C_n^{q_n}(f;x)-f(x)\right|}{1+x^2}\leqslant\frac{\varepsilon}{\sqrt{b_n}}+C_f(\delta)\left[\frac{3\sqrt{b_n}}{[n]_{q_n}}+\frac{b_n^{3/2}}{[n]_{q_n}^2}+\sqrt{\frac{3b_n}{[n]_{q_n}}+\frac{b_n}{[n]_{q_n}^2}}\right]$$

which proves the theorem. □

4. Voronovskaja type theorem

Now, we give a Voronovskaja type theorem for $C_n^q(f,x)$.

Lemma 4.1. *Let $q:=(q_n)$, $0<q_n\leqslant 1$, be sequence such that $q_n\longrightarrow 1$ as $n\longrightarrow\infty$. Then, we have the following limits:*

(i) $\lim_{n\to\infty}\frac{[n]_{q_n}}{b_n}C_n^{q_n}((t-x)^2,x)=x$

(ii) $\lim_{n\to\infty}\frac{[n]_{q_n}^2}{b_n^2}C_n^{q_n}((t-x)^4,x)=2x^2$.

Proof. (i) From Lemma 2.3, we have

$$C_n^{q_n}((t-x)^2;x) = -\frac{x^2}{[n]_{q_n}} + \frac{b_n}{[n]_{q_n}}\frac{q_n^3+2q_n^2+3q_n}{q_n^3+2q_n^2+2q_n+1}x + \frac{b_n^2}{(q_n^2+q_n+1)[n]_{q_n}^2}. \quad (4.1)$$

Then, we get

$$\frac{[n]_{q_n}}{b_n}C_n^{q_n}((t-x)^2,x) = -\frac{x^2}{b_n} + \frac{q_n^3+2q_n^2+3q_n}{q_n^3+2q_n^2+2q_n+1}x + \frac{b_n}{(q_n^2+q_n+1)[n]_{q_n}}.$$

Let us take the limit of both sides of the above equality as $n \longrightarrow \infty$, then we have

$$\begin{aligned}\lim_{n\to\infty}\frac{[n]_{q_n}}{b_n}\left\{C_n^{q_n}((t-x)^2,x)\right\} &= \lim_{n\to\infty}\left\{-\frac{x^2}{b_n} + \frac{q_n^3+2q_n^2+3q_n}{q_n^3+2q_n^2+2q_n+1}x\right. \\ &\quad \left. + \frac{b_n}{(q_n^2+q_n+1)[n]_{q_n}}\right\} \\ &= x.\end{aligned}$$

(ii) Again from Lemma 2.3 and by the linearity of the operators $C_n^{q_n}(f,x)$, we get

$$C_n^{q}((t-x)^4,x) = D_{1,n}x^4 + D_{2,n}x^3 + D_{3,n}x^2 + D_{4,n}x + D_{5,n},$$

where

$$\begin{aligned}D_{1,n} &= \frac{q_n^6[n-1]_{q_n}[n-2]_{q_n}[n-3]_{q_n}}{[n]_{q_n}^3} - \frac{4q_n^3[n-1]_{q_n}[n-2]_{q_n}}{[n]_{q_n}^2} + \frac{6q_n[n-1]_{q_n}}{[n]_{q_n}} - 3, \\ D_{2,n} &= \left\{\frac{q_n^3[n-1]_{q_n}[n-2]_{q_n}}{[n]_{q_n}^2}K_{5,q_n} - \frac{4K_{3,q_n}q_n[n-1]_{q_n}}{[n]_{q_n}} + 6K_{2,q_n} - 4K_{0,q_n}\right\}\frac{b_n}{[n]_{q_n}}, \\ D_{3,n} &= \left\{\frac{q_n[n-1]_{q_n}}{[n]_{q_n}}K_{6,q_n} + 6K_{1,q_n} - 4K_{4,q_n}\right\}\frac{b_n^2}{[n]_{q_n}^2}, \\ D_{4,n} &= \left\{K_{7,q_n} - \frac{4}{q_n^3+q_n^2+q_n+1}\right\}\frac{b_n^3}{[n]_{q_n}^3}, \\ D_{5,n} &= \frac{1}{q^4+q_n^3+q_n^2+q_n+1}\frac{b_n^4}{[n]_{q_n}^4}.\end{aligned}$$

and

$$\begin{aligned}
K_{0,q_n} &= \frac{1}{1+q_n},\\
K_{1,q_n} &= \frac{q_n^2+3q_n+2}{q_n^2+q_n+1},\\
K_{2,q_n} &= \frac{1}{q_n^2+q_n+1}\\
K_{3,q_n} &= \frac{q_n^4+3q_n^3+6q_n^2+5q_n+3}{q_n^3+q_n^2+q_n+1},\\
K_{4,q_n} &= \frac{q_n^3+4q_n^2+6q_n+2}{q_n^3+q_n^2+q_n+1},\\
K_{5,q_n} &= \frac{q_n^6+3q_n^5+6q_n^4+10q_n^3+9q_n^2+7q_n+4}{q_n^4+q_n^3+q_n^2+q_n+1},\\
K_{6,q_n} &= \frac{q_n^6+4q_n^5+11q_n^4+18q_n^3+15q_n^2+11q_n+5}{q_n^4+q_n^3+q_n^2+q_n+1},\\
K_{7,q_n} &= \frac{q_n^4+5q_n^3+10q_n^2+10q_n+4}{q_n^4+q_n^3+q_n^2+q_n+1}.
\end{aligned}$$

By (2.1), we get

$$\begin{aligned}
&\lim_{n\to\infty}\frac{[n]_{q_n}^2}{b_n^2}\{D_{1,n}\}\\
&= \lim_{n\to\infty}\frac{[n]_{q_n}^2}{b_n^2}\left\{\frac{-(1-q_n)^2[n]_{q_n}^2+[n]_{q_n}(q_n^3+3q_n^2-1)-(q_n^3+q_n^2+2q_n+1)}{[n]_{q_n}^3}\right\}\\
&= \lim_{n\to\infty}\left\{\frac{-(1-q_n)^2[n]_{q_n}}{b_n^2}+\frac{q_n^3+3q_n^2-1}{b_n^2}-\frac{q_n^3+q_n^2+2q_n+1}{[n]_{q_n}b_n^2}\right\}\\
&= \lim_{n\to\infty}\left\{\frac{(q_n-1)(1-q_n^n)}{b_n^2}+\frac{q_n^3+3q_n^2-1}{b_n^2}-\frac{q_n^3+q_n^2+2q_n+1}{[n]_{q_n}b_n}\right\}=0. \qquad (4.2)
\end{aligned}$$

Again, by using (2.1), we have

$$\frac{[n]_{q_n}^2}{b_n^2}\{D_{2,n}\}=\frac{[n]_{q_n}}{b_n}\left\{\frac{[n]_{q_n}^2(K_{5,q_n}-4K_{3,q_n}+6K_{2,q_n}-4K_{0,q_n})+[n]_{q_n}(4K_{3,q_n}-q_n-2)(q_n+1)K_{6,q_n}}{[n]_{q_n}^2}\right\}$$

Taking the limit of both sides of the above equality, we get

$$\begin{aligned}
\lim_{n\to\infty}\frac{[n]_{q_n}^2}{b_n^2}\{D_{2,n}\} &= \lim_{n\to\infty}\left\{\frac{[n]_{q_n}(K_{5,q_n}-4K_{3,q_n}+6K_{2,q_n}-4K_{0,q_n})}{b_n}\right.\\
&\qquad\left.+\frac{4K_{3,q_n}-q_n-2}{b_n}+\frac{(q_n+1)K_{6,q_n}}{[n]_{q_n}b_n}\right\}
\end{aligned}$$

Since $\lim_{n\to\infty}\frac{4K_{3,q_n}-q_n-2}{b_n}=0$ and $\lim_{n\to\infty}\frac{(q_n+1)K_{6,q_n}}{[n]_{q_n}b_n}=0$, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{[n]_{q_n}^2}{b_n^2}\{D_{2,n}\} &= \lim_{n\to\infty}\left\{\frac{[n]_{q_n}(K_{5,q_n}-4K_{3,q_n}+6K_{2,q_n}-4K_{0,q_n})}{b_n}\right\}\\ &= \lim_{n\to\infty}\left\{\frac{q_n^2(1-q_n)(1-q_n^n)}{(q_n+1)(q_n^3+q_n^2+q_n)}\right.\\ &\quad\left.\cdot\frac{(q_n^8+3q_n^7+6q_n^6+9q_n^5+12q_n^4+9q_n^3+8q_n^2+7q_n+5)}{(q_n^5+q_n^4+q_n^3+q_n^2+q_n+1)b_n}\right\}\\ &= 0. \end{aligned}\tag{4.3}$$

Finally, using (2.1), we get

$$\begin{aligned}\lim_{n\to\infty}\frac{[n]_{q_n}^2}{b_n^2}\{D_{3,n}\} &= \lim_{n\to\infty}\left\{-\frac{K_{6,q_n}}{[n]_{q_n}}+K_{6,q_n}+6K_{1,q_n}-4K_{4,q_n}\right\}\\ &= K_{6,q_n}+6K_{1,q_n}-4K_{4,q_n}=2. \end{aligned}\tag{4.4}$$

It is clear that

$$\lim_{n\to\infty}\frac{[n]_{q_n}^2}{b_n^2}\{D_{4,n}x+D_{5,n}\}=0.$$

By combining (4.2)-(**??**), we reach the desired the result. □

Theorem 4.2. *Let $f\in C_{x^2}^*[0,\infty)$ such that $f',f''\in C_{x^2}^*[0,\infty)$. Then, we have*

$$\lim_{n\to\infty}\frac{[n]_{q_n}}{b_n}\left(C_n^{q_n}(f,x)-f(x)\right)=\frac{1}{2}f'(x)+\frac{x}{2}f''(x).$$

Proof. We write Taylor's expansion of f as follows:

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+\varepsilon(t,x)(t-x)^2,$$

where $\varepsilon(t,x)\longrightarrow 0$ as $t\longrightarrow x$. By linearity of the operators $C_n^{q_n}(f,x)$ we get

$$C_n^{q_n}(f,x)-f(x)=f'(x)C_n^{q_n}((t-x),x)+\frac{1}{2}f''(x)C_n^{q_n}((t-x)^2,x)+C_n^{q_n}\left(\varepsilon(t,x)(t-x)^2,x\right).$$

From Lemma 2.3, we have □

$$\begin{aligned}C_n^{q_n}(f,x)-f(x) &= f'(x)\frac{b_n}{(1+q_n)[n]_{q_n}}\\ &\quad+\frac{1}{2}f''(x)\left(-\frac{x^2}{[n]_{q_n}}+\frac{b_n}{[n]_{q_n}}\frac{q_n^3+2q_n^2+3q_n}{q_n^3+2q_n^2+2q_n+1}x+\frac{b_n^2}{(q_n^2+q_n+1)[n]_{q_n}^2}\right)\\ &\quad+C_n^{q_n}\left(\varepsilon(t,x)(t-x)^2,x\right)\end{aligned}$$

For the last term on the right hand side, using Cauchy-Schwartz inequality, we get

$$\begin{aligned}\lim_{n\to\infty}\frac{[n]_{q_n}}{b_n}C_n^{q_n}\left(\varepsilon(t,x)(t-x)^2,x\right) &\leqslant \sqrt{\lim_{n\to\infty}C_n^{q_n}\left(\varepsilon^2(t,x),x\right)}\\ &\cdot\sqrt{\lim_{n\to\infty}\frac{[n]_{q_n}^2}{b_n^2}C_n^{q_n}\left((t-x)^4,x\right)}.\end{aligned}$$

Since $\lim_{n\to\infty}C_n^{q_n}\left(\varepsilon^2(t,x),x\right)=0$ and by Lemma 4.1(ii) $\lim_{n\to\infty}\frac{[n]_{q_n}^2}{b_n^2}C_n^{q_n}\left((t-x)^4,x\right)$ is finite, we have $\lim_{n\to\infty}\frac{[n]_{q_n}}{b_n}C_n^{q_n}\left(\varepsilon(t,x)(t-x)^2,x\right)=0$. Therefore, we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{[n]_{q_n}}{b_n}\left(C_n^{q_n}(f,x)-f(x)\right) &= \frac{1}{2}f''(x)\lim_{n\to\infty}\left(-\frac{x^2}{b_n}+\frac{q_n^3+2q_n^2+3q_n}{q_n^3+2q_n^2+2q_n+1}x\right.\\ &\left.+\frac{b_n}{(q_n^2+q_n+1)[n]_{q_n}}\right)+\frac{f'(x)}{1+q_n}\\ &= \frac{1}{2}f'(x)+\frac{x}{2}f''(x).\end{aligned}$$

This step completes the proof.

5. Local approximation

In this section, we give a local approximation theorem regarding the our operators. The Peetre's K-functional is defined by

$$K_2(f;\delta):=\inf_{g\in C^2[0,\infty)}\left\{\|f-g\|+\delta\|g''\|\right\},$$

where $C_B^2[0,\infty)=\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\}$, $C_B[0,\infty)$ denotes the space of all real-valued bounded and continuous functions.

By using Devore-Lorentz theorem (see[19], Thm 2.4, pp.177), for $f\in C_B[0,\infty)$ and $C>0$ we have

$$K_2(f;\delta)\leqslant C\omega_2\left(f;\sqrt{\delta}\right) \tag{5.1}$$

where ω_2 is the second modulus of continuity of f.

In this section, we need the following lemmas for proving our main theorem.

Lemma 5.1. *Let $g\in C_B^2[0,\infty)$. The following inequality holds:*

$$\left|\tilde{C}_n^q(g;x)-g(x)\right|\leq\Theta_n(x)\|g''\|,$$

where $\Theta_n(x)=\frac{x(3b_n-x)}{[n]_q}+\frac{b_n^2}{[n]_q^2}+\frac{b_n}{[n]_q}$.

Proof. Let us define auxiliary operators

$$\tilde{C}_n^q(f;x):=C_n^q(f;x)-f\left(x+\frac{1}{1+q}\frac{b_n}{[n]_q}\right)+f(x).$$

It is easy to see that $\tilde{C}_n^q\left(t-x;x\right)=0$. Let $g\in C_B^2[0,\infty)$. By using Taylor expansion of g, we obtain

$$g\left(t\right)-g\left(x\right)=\left(t-x\right)g'\left(x\right)+\int_x^t\left(t-u\right)g''\left(u\right)du$$

Applying the operator $\tilde{C}_n^q$ to the above equality, we get

$$\begin{aligned}\tilde{C}_n^q\left(g;x\right)-g\left(x\right) &= g'\left(x\right)\tilde{C}_n^q\left(t-x;x\right)-\tilde{C}_n^q\left(\int_x^t\left(t-u\right)g^{''}\left(u\right)du;x\right)\\ &= \tilde{C}_n^q\left(\int_x^t\left(t-u\right)g^{''}\left(u\right)du;x\right)\\ &= C_n^q\left(\int_x^t\left(t-u\right)g''\left(u\right)du;x\right)\\ &\quad -\int_x^{x+\frac{1}{1+q}\frac{b_n}{[n]_q}}\left(x+\frac{1}{1+q}\frac{b_n}{\left[n\right]_q}-u\right)g^{''}\left(u\right)du.\end{aligned}$$

Thus, we have

$$\begin{aligned}\left|\tilde{C}_n^q\left(g;x\right)+g\left(x\right)\right| &\leqslant C_n^q\left(\left|\int_x^t\left(t-u\right)g^{''}\left(u\right)du\right|;x\right)\\ &\quad +\left|\int_x^{x+\frac{1}{1+q}\frac{b_n}{[n]_q}}\left(x+\frac{1}{1+q}\frac{b_n}{\left[n\right]_q}-u\right)g^{''}\left(u\right)du\right|.\end{aligned}$$

Since

$$\left|\int_x^t\left(t-u\right)g^{''}\left(u\right)du\right|\leqslant\left(t-x\right)^2\left\|g^{''}\right\|$$

we get

$$\left|\int_x^{x+\frac{1}{1+q}\frac{b_n}{[n]_q}}\left(x+\frac{1}{1+q}\frac{b_n}{\left[n\right]_q}-u\right)g^{''}\left(u\right)du\right|\leqslant\left(\frac{1}{1+q}\frac{b_n}{\left[n\right]_q}\right)^2\left\|g^{''}\right\|.$$

We can write

$$\left|\tilde{C}_n^q\left(g;x\right)-g\left(x\right)\right|\leqslant\left\{C_n^q\left(\left(t-x\right)^2;x\right)+\left(\frac{1}{1+q}\frac{b_n}{\left[n\right]_q}\right)^2\right\}\left\|g^{''}\right\|.$$

Then, by using Lemma 2.3, we may write

$$\left|\tilde{C}_n^q\left(g;x\right)-g\left(x\right)\right|\leqslant\Theta_n(x)\left\|g''\right\|.$$

□

Theorem 5.2. *If $f\in C_B\left[0,\infty\right)$ we have*

$$\left|C_n^q(f;x)-f(x)\right|\leqslant C\omega_2\left(f,\frac{1}{2}\sqrt{\Theta_n(x)}\right)+\omega\left(f;\frac{b_n}{[n]_q}\right),$$

where $C>0$ is a constant.

Proof. Assume that $f\in C_B[0,\infty)$ by using the definition of $\tilde{C}_n^q(f;x)$, we get

$$\begin{aligned}\left|C_n^q(f;x)-f(x)\right| &\leqslant \left|\tilde{C}_n^q(f-g;x)\right|+\left|(f-g)(x)\right|+\left|\tilde{C}_n^q(g;x)-g(x)\right|\\ &+ \left|f\left(x+\frac{1}{1+q}\frac{b_n}{[n]_q}\right)-f(x)\right|\end{aligned}$$

and

$$\left|\tilde{C}_n^q(f;x)\right|\leqslant\|f\|\,C_n^q(1;x)+2\|f\|=3\|f\|.$$

Thus, we obtain

$$\left|C_n^q(f;x)-f(x)\right|\leqslant 4\|f-g\|+\left|\tilde{C}_n^q(g;x)-g(x)\right|+\omega\left(f;\frac{b_n}{[n]_q}\right)$$

and using Lemma 5.1,

$$\left|C_n^q(f;x)-f(x)\right|\leqslant 4\left(\|f-g\|+\frac{1}{4}\Theta_n(x)\|g''\|\right)+\omega\left(f;\frac{b_n}{[n]_q}\right).$$

Taking the infimum over all $g\in C_B^2[0,\infty)$ on the right hand side of above inequality and using (5.1), the proof is finished. □

6. Rete of convergence in weighted space

We know that usual first modulus of continuity $\omega(\delta)$ does not tend to zero, as $\delta\to 0$, on infinite interval. Thus we use weighted modulus of continuity $\Omega(f,\delta)$ defined on infinite interval $\mathbb{R}^+$ (see [18]). Let

$$\mathbf{\Omega}(f,\delta)=\sup_{|h|<\delta,\ x\in\mathbb{R}^+}\frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}\text{ for each } f\in C_{x^2}[0,\infty).$$

Now some elementary properties of $\Omega(f,\delta)$ are collected in the following Lemma.

Lemma 6.1. *Let $f\in C_{x^2}^k[0,\infty)$. Then,*

i) $\Omega(f,\delta)$ is a monotonically increasing function of δ, $\delta\geqslant 0$.
ii) For every $f\in C_{x^2}^*[0,\infty)$, $\lim\limits_{\delta\to 0}\Omega(f,\delta)=0$.
iii) For each $\lambda>0$,

$$\Omega(f,\lambda\delta)\leqslant 2(1+\lambda)\left(1+\delta^2\right)\Omega(f,\delta). \tag{6.1}$$

From the inequality (6.1) and definition of $\Omega(f,\delta)$ we get

$$|f(t)-f(x)|\leqslant 2\left(1+x^2\right)\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta}\right)\left(1+\delta^2\right)\Omega(f,\delta) \tag{6.2}$$

for every $f \in C_{x^2}[0,\infty)$ and $x,t \in \mathbb{R}^+$.

Theorem 6.2. *Let $0 < q \leqslant 1$ and $f \in C^*_{x^2}[0,\infty)$. Then, we have*

$$\sup_{0\leqslant x\leqslant b_n} \frac{|C_n^q(f,x) - f(x)|}{(1+x^2)^3} \leqslant C\Omega\left(f, \sqrt{\frac{b_n}{[n]_{q_n}}}\right)$$

where C is an absolute constant.

Proof. Using (6.2) , we get

$$\begin{aligned} |C_n^q(f,x) - f(x)| &= C_n^q(|f(t) - f(x)|, x) \\ &\leqslant 2\left(1+x^2\right)\left(1+\delta^2\right) C_n^q\left(\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta}\right), x\right)\Omega(f,\delta) \end{aligned}$$

also we can write that

$$\begin{aligned} & C_n^q\left(\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta}\right), x\right) \\ = {} & 1 + C_n^q\left((t-x)^2; x\right) + \frac{1}{\delta} C_n^q(|t-x|; x) + \frac{1}{\delta} C_n^q\left(|t-x|(t-x)^2; x\right) \\ \leqslant {} & 1 + C_n^q\left((t-x)^2; x\right) + \frac{1}{\delta}\sqrt{C_n^q\left((t-x)^2; x\right)} \\ & + \frac{1}{\delta}\sqrt{C_n^q\left((t-x)^2; x\right)}\sqrt{C_n^q\left((t-x)^4; x\right)}. \end{aligned}$$

From (4.1) and (**??**), we know that

$$C_n^q\left((t-x)^2; x\right) = \mathcal{O}\left(\frac{b_n}{[n]_{q_n}}\right)\left(x^2+x+1\right)$$

and

$$C_n^q\left((t-x)^4; x\right) = \mathcal{O}\left(\frac{b_n^2}{[n]_{q_n}^2}\right)\left(x^4+x^3+x^2+x+1\right).$$

Choosing $\delta = \sqrt{\frac{b_n}{[n]_{q_n}}}$ we have

$$\begin{aligned} & |C_n^q(f,x) - f(x)| \\ \leqslant {} & 2\left(1+x^2\right)\left(1+\delta^2\right) C_n^q\left(\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta}\right), x\right)\Omega(f,\delta) \\ \leqslant {} & 4\left(1+x^2\right)\Omega\left(f, \sqrt{\frac{b_n}{[n]_{q_n}}}\right) \end{aligned}$$

$$\times\left(1 + \mathcal{O}\left(\frac{b_n}{[n]_{q_n}}\right)\left(x^2+x+1\right) + \sqrt{(x^2+x+1)}\sqrt{\mathcal{O}\left(\frac{b_n^2}{[n]_{q_n}^2}\right)\left(x^4+x^3+x^2+x+1\right)}\right),$$

which proves the theorem. □

7. Statistical Convergence of $C_n^q(f;x)$

In 1951, Fast [14] and Steinhaus [22] defined the notion of statistical convergence for sequences of real numbers as:

Let M be a subset of the set of natural numbers $\mathbb{N}$. Then, $M_n = \{k \leqslant n : k \in M\}$. The natural density of M is defined by $\delta(M) = \lim_n \frac{1}{n}|M_n|$ provided that the limit exists, where $|M_n|$ denotes the cardinality of the set M_n. A sequence $x = (x_k)$ is called statistically convergent to the number $\ell \in \mathbb{R}$, denoted by $st - \lim x = \ell$. For each $\epsilon > 0$, the set $M_\varepsilon = \{k \in \mathbb{N} : |x_k - \ell| \geqslant \epsilon\}$ has a natural density zero, that is

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leqslant n : |x_k - \ell| \geqslant \epsilon\}| = 0.$$

This concept was used in approximation theory by Gadjiev and Orhan [16] in 2002. They proved the Bohman–Korovkin type approximation theorem [12] for statistical convergence. Currently, researchers studying statistical convergence have devoted their effort to statistical approximation.

In this section, we examine the statistical approximation properties of the $C_n^q(f;x)$.

Theorem 7.1. *$q := (q_n)$, $0 < q_n \leqslant 1$ be a sequence satisfying the following conditions:*

$$st - \lim_{n\to\infty} q_n = 1,\ st - \lim_{n\to\infty} q_n^n = a,\quad st - \lim_{n\to\infty} \frac{b_n}{[n]_{q_n}} = 0. \tag{7.1}$$

Let f be a monotone increasing function on $[0,\infty)$ then,

$$st - \lim_{n\to\infty} \|C_n^{q_n}(f) - f\|_{x^2} = 0.$$

Proof. Since $C_n^q(f;x)$ is a linear-positive operator, if we show that $st - \lim_{n\to\infty} \|C_n^{q_n}(e_i) - e_i\|_{x^2} = 0$, where $e_i = x^i, i = 0,1,2,$ then we are done. It is clear that

$$st - \lim_{n\to\infty} \|C_n^{q_n}(e_0) - e_0\|_{x^2} = 0. \tag{7.2}$$

Using (2.2), we get

$$(C_n^{q_n}(e_1) - e_1) = \frac{1}{1+q_n}\frac{b_n}{[n]_{q_n}}.$$

Let $\varepsilon > 0$, then we define the following set:

$$K := \{k : \|C_n^{q_k}(e_1) - e_1\|_{x^2} \geqslant \varepsilon\} = \left\{k : \frac{1}{1+q_k}\frac{b_k}{[k]_{q_k}} \geqslant \varepsilon\right\}.$$

One can obtain that

$$\delta\{k \leqslant n : \|C_n^{q_k}(e_1) - e_1\|_{x^2} \geqslant \varepsilon\} \leqslant \delta\left\{k \leqslant n : \frac{1}{1+q_k}\frac{b_k}{[k]_{q_k}}\right\}.$$

Thus, we get

$$st - \lim_{n\to\infty} \|C_n^{q_n}(e_1) - e_1\|_{x^2} = 0. \tag{7.3}$$

Using (2.3), we can write

$$(C_n^{q_n}(e_2)-e_2) = -\frac{x^2}{[n]_{q_n}} + x\frac{q_n^2+3q_n+2}{q_n^2+q_n+1}\frac{b_n}{[n]_{q_n}} + \frac{1}{q_n^2+q_n+1}\frac{b_n^2}{[n]_{q_n}^2}.$$

Thus, we get

$$\|C_n^{q_n}(e_2)-e_2\|_{x^2} \leqslant \frac{1}{[n]_{q_n}}\|e_2\|_{x^2} + \frac{q_n^2+3q_n+2}{q_n^2+q_n+1}\frac{b_n}{[n]_q}\|e_1\|_{x^2} + \frac{1}{q_n^2+q_n+1}\frac{b_n^2}{[n]_q^2}.$$

Let $\varepsilon > 0$, we define the following sets:

$$\begin{aligned}
U &:= \left\{k := \|C_n^{q_k}(e_2)-e_2\|_{x^2} \geqslant \varepsilon\right\},\\
U_1 &:= \left\{k := \frac{a^2}{[k]_{q_k}} \geqslant \frac{\varepsilon}{3}\right\},\\
U_2 &:= \left\{k := a\frac{q_k^2+3q_k+2}{(q_k^2+q_k+1)}\frac{b_k}{[k]_{q_k}} \geqslant \frac{\varepsilon}{3}\right\},\\
U_3 &:= \left\{k := \frac{b_k^2}{(q_k^2+q_k+1)[k]_{q_k}^2} \geqslant \frac{\varepsilon}{3}\right\}.
\end{aligned}$$

It is clear that $U \subseteq U_1 \cup U_2 \cup U_3$. Thus,

$$\begin{aligned}
\delta\left\{k \leqslant n : \|C_n^{q_k}(e_2)-e_2\|_{x^2} \geqslant \varepsilon\right\} \leqslant\ & \delta\left\{k \leqslant n : \frac{a^2}{[k]_{q_k}} \geqslant \frac{\varepsilon}{3}\right\}\\
&+\delta\left\{k \leqslant n : a\frac{q_k^2+3q_k+2}{(q_k^2+q_k+1)}\frac{b_k}{[k]_{q_k}} \geqslant \frac{\varepsilon}{3}\right\}\\
&+\delta\left\{k \leqslant n : \frac{b_k^2}{(q_k^2+q_k+1)[k]_{q_k}^2} \geqslant \frac{\varepsilon}{3}\right\}
\end{aligned}$$

and since (q_n) satisfies (7.1), we obtain

$$st - \lim_{n\to\infty} \|C_n^{q_n}(e_2)-e_2\|_{x^2} = 0. \tag{7.4}$$

Hence, by using statistical Korovkin theorem, the desired result follows from (7.2)-(7.4). □

Note: It is obvious that $f(x) \geqslant 0$ does not guarantee the positivity of the operators $C_n^q(f;x)$. Thus, we assumed that f is a monotone increasing function on $[0,b_n]$. By using this assumption, we showed the statistical convergence of the operators via Korovkin theorem. However, this assumption is not sufficient to investigate the rate of convergence and order of approximation because of the usual definition of $q-$integral. In order to solve this problem, there are two ways proposed by Gauchman[17] and Marinkovic [20]. They defined different types of $q-$integrals namely, restricted $q-$integral and Riemann type $q-$integral respectively.

In this study, we redefine $q-$Chlodowsky-Kantorovich operators by using Riemann type $q-$integral.

Definition 1. *(Riemann type $q-$integral) Let $0 < q < 1$ and $0 < a < b$. The Riemann type $q-$integral is defined as follows:*

$$\int_a^b f(x)\ d_q^R x = (1-q)(b-a)\sum_{j=0}^{\infty} f\left(a+(b-a)q^j\right)q^j.$$

The modified version of $C_n^q(f;x)$ via Riemann type $q-$integral is as follows:

$$\hat{C}_n^q(f;x) = [n]_q \sum_{k=0}^{n} q^{-k} \left[\begin{array}{c} n \\ k \end{array}\right]_q \left(\frac{x}{b_n}\right)^k \left(1-\frac{x}{b_n}\right)_q^{n-k} \int_{[k]_q/[n]_q}^{[k+1]_q/[n]_q} f(b_n t)\, d_q^R t.$$

Lemma 7.2. *Let $0 < q < 1$ and $0 < a < b$, $\frac{1}{m}+\frac{1}{n}=1$, for $R_q(|fg|;a,b) \leq (R_q(|f|^m;a,b))^{1/m}(R_q(|g|^m;a,b))^{1/n}$, where $R_q(f;a,b)=\int_a^b f(x)\ d_q^R x$.*

Proof. : Given in [13]. □

Remark 7.3. *From ([13]), we can obtain the following integrals via making necessary computations .*

$$\begin{aligned}
\int_{[k]_q/[n]_q}^{[k+1]_q/[n]_q} d_q^R t &= \frac{q^k}{[n]_q}, \\
\int_{[k]_q/[n]_q}^{[k+1]_q/[n]_q} b_n t\, d_q^R t &= q^k [k]_q \frac{b_n}{[n]_q^2} + \frac{q^{2k}}{[2]_q}\frac{b_n}{[n]_q^2}, \\
\int_{[k]_q/[n]_q}^{[k+1]_q/[n]_q} b_n^2 t^2\, d_q^R t &= q^k [k]_q^2 \frac{b_n^2}{[n]_q^3} + \frac{2q^{2k}[k]_q}{[2]_q}\frac{b_n^2}{[n]_q^3} + \frac{q^{3k}}{[3]_q}\frac{b_n^2}{[n]_q^3}.
\end{aligned}$$

Lemma 7.4. *By using the above $q-$Riemann type integrals, we can obtain the following formulas for the moments of $\hat{C}_n^q(f;x)$:*

$$\begin{aligned}
\hat{C}_n^q(1;x) &= 1, \\
\hat{C}_n^q(t;x) &= \frac{2q}{1+q}x + \frac{b_n}{[n]_q}, \\
\hat{C}_n^q(t^2;x) &= q^2\left(\frac{4q^2+q+1}{q^3+2q^2+2q+1}\right)\frac{[n-1]_q}{[n]_q}x^2 + q\left(\frac{4q^2+5q+3}{q^3+2q^2+2q+1}\right)\frac{b_n}{[n]_q}x \\
&\quad + \frac{1}{q^2+q+1}\frac{b_n^2}{[n]_q^2}.
\end{aligned}$$

Now, we can give the statistical approximation of $\hat{C}_n^q(f;x)$ in the following theorem.

Theorem 7.5. *Let $q := (q_n)$ be sequence satisfying* (7.1) *and let f be a function defined on $[0, \infty)$ by $f \in C_{x^2}[0, \infty)$, then we have*

$$st - \lim_{n\to\infty} \left\| \hat{C}_n^{q_n}(f) - f \right\|_{x^2} = 0.$$

Proof. It is clear that

$$st - \lim_{n\to\infty} \left\| \hat{C}_n^{q_n}(e_0) - e_0 \right\|_{x^2} = 0. \tag{7.5}$$

Secondly,

$$\| \hat{C}_n^{q_n}(e_1) - e_1 \|_{x^2} = \frac{q_n - 1}{q_n + 1} \|e_1\|_{x^2} + \frac{b_n}{[n]_{q_n}} \tag{7.6}$$

Now, for a given $\varepsilon > 0$, we define the following sets:

$$L := \left\{ k : \left\| \hat{C}_n^{q_k}(e_1) - e_1 \right\|_{x^2} \geqslant \varepsilon \right\}$$

$$L_1 := \left\{ k : \frac{q_k - 1}{q_k + 1} \geqslant \frac{\varepsilon}{2} \right\},\ L_2 := \left\{ k : \frac{b_k}{[k]_{q_k}} \geqslant \frac{\varepsilon}{2} \right\}.$$

From (7.6), we see that $L \subseteq L_1 \cup L_2$. So, we get,

$$\delta \left\{ k \leqslant n : \left\| \hat{C}_n^{q_k}(e_1) - e_1 \right\|_{x^2} \geqslant \varepsilon \right\} \leqslant \delta \left\{ k \leqslant n : \frac{q_k - 1}{q_k + 1} \geqslant \frac{\varepsilon}{2} \right\} + \delta \left\{ k \leqslant n : \frac{b_k}{[n]_{q_k}} \geqslant \frac{\varepsilon}{2} \right\}.$$

Since $st - \lim_{n\to\infty} \frac{q_n - 1}{q_n + 1} = 0$ and $st - \lim_{n\to\infty} \frac{b_n}{[n]_{q_n}} = 0$, we have

$$st - \lim_{n\to\infty} \left\| \hat{C}_n^{q_n}(e_1) - e_1 \right\|_{x^2} = 0. \tag{7.7}$$

Finally, we have

$$\begin{aligned} \hat{C}_n^{q_n}(e_2; x) - e_2(x) &= \left(\left(\frac{4q_n^4 + q_n^3 + q_n^2}{q_n^3 + 2q_n^2 + 2q_n + 1} \right) \frac{[n-1]_{q_n}}{[n]_{q_n}} - 1 \right) x^2 \\ &\quad + \left(\frac{4q_n^3 + 5q_n^2 + 3q_n}{q_n^3 + 2q_n^2 + 2q + 1} \right) \frac{b_n}{[n]_{q_n}} x + \frac{1}{q_n^2 + q_n + 1} \frac{b_n^2}{[n]_{q_n}^2}. \end{aligned}$$

Using $[n-1]_{q_n} < [n]_{q_n}$,

$$\begin{aligned} \left\| \hat{C}_n^{q_n}(e_2) - e_2 \right\|_{x^2} \leqslant & \left| \frac{4q_n^4 - q_n^2 - 2q_n - 1}{q_n^3 + 2q_n^2 + 2q_n + 1} \right| \| e_2 \|_{x^2} \\ & + \left| \frac{4q_n^3 + 5q_n^2 + 3q_n}{q_n^3 + 2q_n^2 + 2q_n + 1} \right| \frac{b_n}{[n]_{q_n}} \| e_1 \|_{x^2} + \left| \frac{1}{q_n^2 + q_n + 1} \right| \frac{b_n^2}{[n]_{q_n}^2}. \end{aligned}$$

Let

$$\alpha_n := \frac{4q_n^4 - q_n^2 - 2q_n - 1}{q_n^3 + 2q_n^2 + 2q_n + 1} \text{ and } \beta_n := \frac{4q_n^3 + 5q_n^2 + 3q_n}{q_n^3 + 2q_n^2 + 2q_n + 1} \frac{b_n}{[n]_{q_n}},$$

it is easy to see that the followings hold:

$$st-\lim_{n\to\infty}\alpha_n=0,\ st-\lim_{n\to\infty}\beta_n=0 \text{ and } st-\lim_{n\to\infty}\frac{b_n^2}{[n]_{q_n}^2}=0$$

Similarly, for a given $\varepsilon>0$, we define the following sets:

$$N:=\left\{k:\left\|\hat{C}_n^{q_k}(e_2)-e_2\right\|_{x^2}\geqslant\varepsilon\right\},$$

$$N_1:=\left\{k:\alpha_k\geqslant\frac{\varepsilon}{3}\right\},N_2:=\left\{k:\beta_k\geqslant\frac{\varepsilon}{3}\right\},N_3:=\left\{k:\frac{1}{q_k^2+q_k+1}\frac{b_k^2}{[k]_{q_k}^2}\geqslant\frac{\varepsilon}{3}\right\}.$$

It is obtained that $N\subseteq N_1\cup N_2\cup N_3$. So, we may write,

$$\begin{aligned}\delta\left\{k\leqslant n:\left\|\hat{C}_n^{q_k}(e_2;.)-e_2\right\|_{x^2}\geqslant\varepsilon\right\} &\leqslant \delta\left\{k\leqslant n:\alpha_k\geqslant\frac{\varepsilon}{3}\right\}+\delta\left\{k\leqslant n:\beta_k\geqslant\frac{\varepsilon}{3}\right\}\\ &\quad+\delta\left\{k\leqslant n:\frac{1}{q_k^2+q_k+1}\frac{b_k^2}{[k]_{q_k}^2}\geqslant\frac{\varepsilon}{3}\right\}.\end{aligned}$$

Thus, we obtain

$$st-\lim_{n\to\infty}\left\|\hat{C}_n^{q_n}(e_2;.)-e_2\right\|_{x^2}=0. \tag{7.8}$$

The proof is finished using (7.5), (7.7) and (7.8) via statistical Korovkin's theorem. □

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