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Research Article

A New Approach for the Approximations of Solutions to a Common Fixed Point Problem in Metric Fixed Point Theory

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We provide sufficient conditions for the existence of a unique common fixed point for a pair of mappings $T, S : X \to X$, where X is a nonempty set endowed with a certain metric. Moreover, a numerical algorithm is presented in order to approximate such solution. Our approach is different to the usual used methods in the literature.

1. Introduction and Problem Formulation

Let (X, d) be a complete metric space and $T, S : X \to X$ be two given operators. In this paper, we are interested on the problem:

$$\begin{gathered} \text{Find } x \in X \text{ such that} \\ x = Tx, \\ x = Sx. \end{gathered} \tag{1}$$

We provide sufficient conditions for the existence of one and only one solution to (1). Moreover, we present a numerical algorithm in order to approximate such solution. Our approach is different to the existing methods in the literature.

System (1) arises in the study of different problems from nonlinear analysis. For example, when we deal with the solvability of a system of integral equations, such problem can be formulated as a common fixed point problem for a pair of self-mappings $T, S : X \to X$, where T and S are two operators that depend on the considered problem. For some examples in this direction, we refer to [1–5] and references therein.

The most used techniques for the solvability of problem (1) are based on a compatibility condition introduced by Jungck [6]. Such techniques are interesting and can be useful for the solvability of certain problems (see [6–9] and references therein). However, two major difficulties arise in the use of such approach. At first, the compatibility condition is not always satisfied, and in some cases it is not easy to check such condition. Moreover, the numerical approximation of the common fixed point is constructed via the axiom of choice using certain inclusions, which makes its numerical implementation difficult.

In this paper, problem (1) is investigated under the following assumptions.

Assumption (A1). We suppose that X is equipped with a partial order $\preceq$. Recall that $\preceq$ is a partial order on X if it satisfies the following conditions:

(i) $x \preceq x$, for every $x \in X$.

(ii) $x \preceq y$ and $y \preceq z$ imply that $x \preceq z$, for every $(x, y, z) \in X \times X \times X$.

(iii) $x \preceq y$ and $y \preceq x$ imply that $x = y$, for every $(x, y) \in X \times X$.

Assumption (A2). The operator $S : X \to X$ is level closed from the left; that is, the set

$$\operatorname{lev} S_{\preceq} = \{x \in X : x \preceq Sx\} \tag{2}$$

is nonempty and closed.

In order to make the lecture for the reader easy, let us give an example.

Example 1. Let $X = C([0,1];\mathbb{R})$ be the set of real valued and continuous functions on $[0,1]$. We consider the metric d on X defined by

$$d(x,y) = \max\{|x(t) - y(t)| : t \in [0,1]\}, \quad (x,y) \in X \times X. \tag{3}$$

We endow X with the partial order $\preceq$ given by

$$(x,y) \in X \times X, \quad x \preceq y \Longleftrightarrow x(t) \le y(t), \ t \in [0,1]. \tag{4}$$

Next, define the operator $S : X \to X$ by

$$(Sx)(t) = \int_0^t x(s)\,ds, \quad t \in [0,1]. \tag{5}$$

Clearly, $S : X \to X$ is well-defined. Now, consider the set

$$\operatorname{lev} S_{\preceq} = \{x \in X : x \preceq Sx\}, \tag{6}$$

that is,

$$\operatorname{lev} S_{\preceq} = \left\{x \in X : x(t) \le \int_0^t x(s)\,ds, t \in [0,1]\right\}. \tag{7}$$

Let $\{x_n\} \subset \operatorname{lev} S_{\preceq}$ be a sequence that converges to some $x \in X$ (with respect to d); that is,

$$x_n(t) \le \int_0^t x_n(s)\,ds, \quad \forall t \in [0,1], \ \forall n, \qquad d(x_n, x) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{8}$$

Since the uniform convergence implies the point-wise convergence, for all $t \in [0,1]$, we have

$$\lim_{n\to\infty} x_n(t) = x(t). \tag{9}$$

Moreover, for all $t \in [0,1]$,

$$\left|\int_0^t x_n(s)\,ds - \int_0^t x(s)\,ds\right| \le \int_0^t d(x_n,x)\,ds \le d(x_n,x) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{10}$$

Therefore,

$$x(t) \le \int_0^t x(s)\,ds, \quad \forall t \in [0,1], \tag{11}$$

which proves that $S : X \to X$ is level closed from the left.

Remark 2. Note that the fact that $S : X \to X$ is level closed from the left does not imply that

$$\operatorname{lev} S_{\succeq} = \{x \in X : x \succeq Sx\} \tag{12}$$

is closed. Several counterexamples can be obtained. We invite the reader to check this fact by himself.

Assumption (A3). For every $x \in X$, we have

$$x \preceq Sx \Longrightarrow Tx \succeq STx, \qquad x \succeq Sx \Longrightarrow Tx \preceq STx. \tag{13}$$

In order to fix our next assumption, we need to introduce the following class of mappings. We denote by Ψ the set of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the conditions:

(Ψ_1) ψ is nondecreasing.

(Ψ_2) For all $t > 0$, we have

$$\mu_0(t) := \sum_{k=0}^{\infty} \psi^k(t) < \infty. \tag{14}$$

Here, ψ^k is the kth iterate of ψ. Any function $\psi \in \Psi$ is said to be a (c)-comparison function.

We have the following properties of (c)-comparison functions.

Lemma 3 (see [10]). *Let $\psi \in \Psi$. Then*

(i) *$\psi(t) < t$, for all $t > 0$,*

(ii) *$\psi(0) = 0$,*

(iii) *ψ is continuous at $t = 0$,*

(iv) *μ_0 is nondecreasing and continuous at* 0.

Our next assumption is the following.

Assumption (A4). There exists a function $\psi \in \Psi$ such that, for every $(x,y) \in X \times X$, we have

$$x \preceq Sx, \qquad y \succeq Sy \Longrightarrow d(Tx, Ty) \le \psi(d(x,y)). \tag{15}$$

Now, we are ready to state and prove our main result.

2. A Common Fixed Point Result and Approximations

Our main result is given by the following theorem.

Theorem 4. *Suppose that Assumptions (A1)–(A4) are satisfied. Then*

(i) *for any $x_0 \in \operatorname{lev} S_{\preceq}$, the Picard sequence $\{T^n x_0\}$ converges to some $x^* \in X$, which is a solution to (1),*

(ii) *$x^* \in X$ is the unique solution to (1),*

(iii) *the following estimates*

$$d(T^n x_0, x^*) \le \mu_n(d(Tx_0, x_0)), \quad n = 0, 1, 2, \ldots, \tag{16}$$

$$d(T^n x_0, x^*) \le \mu_1\left(d\left(T^{n-1}x_0, T^n x_0\right)\right), \quad n = 1, 2, 3, \ldots \tag{17}$$

hold, where

$$\mu_n(t) = \sum_{k=n}^{\infty} \psi^k(t), \quad t \ge 0, \ n = 0, 1, 2, \ldots. \tag{18}$$

Proof. Let x_0 be an arbitrary element of $\operatorname{lev} S_{\preceq}$; that is,

$$\begin{gathered} x_0 \in X, \\ x_0 \preceq Sx_0. \end{gathered} \tag{19}$$

Such an element exists from Assumption (A2). From Assumption (A3), we have

$$x_1 \succeq Sx_1, \tag{20}$$

where $x_1 = Tx_0$. Again, from Assumption (A3), we have

$$x_2 \preceq Sx_2, \tag{21}$$

where $x_2 = Tx_1$. Now, let us consider the Picard sequence $\{x_n\} \subset X$ defined by

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots. \tag{22}$$

Proceeding as above, by induction we get

$$\begin{gathered} x_{2n} \preceq Sx_{2n}, \\ x_{2n+1} \succeq Sx_{2n+1}, \\ n = 0, 1, 2, \ldots. \end{gathered} \tag{23}$$

Therefore, by Assumption (A4), we have

$$d(Tx_{2n}, Tx_{2n+1}) \le \psi(d(x_{2n}, x_{2n+1})), \quad n = 0, 1, 2, \ldots. \tag{24}$$

Again, by Assumption (A4), we have

$$d(Tx_{2n+1}, Tx_{2n+2}) \le \psi(d(x_{2n+1}, x_{2n+2})), \quad n = 0, 1, 2, \ldots. \tag{25}$$

As a consequence, we have

$$d(x_{n+1}, x_n) \le \psi(d(x_n, x_{n-1})), \quad n = 1, 2, 3, \ldots. \tag{26}$$

From (26), since ψ is a nondecreasing function, for every $n = 1, 2, 3, \ldots$, we have

$$\begin{aligned} d(x_{n+1}, x_n) &\le \psi(d(x_n, x_{n-1})) \le \psi^2(d(x_{n-1}, x_{n-2})) \\ &\le \cdots \le \psi^n(d(x_1, x_0)). \end{aligned} \tag{27}$$

Suppose that

$$d(x_1, x_0) = 0. \tag{28}$$

In this case, from (23), we have

$$\begin{gathered} x_0 = x_1 = Tx_0, \\ x_0 \preceq Sx_0, \\ x_0 = x_1 \succeq Sx_1 = Sx_0. \end{gathered} \tag{29}$$

Since $\preceq$ is a partial order, this proves that $x_0 \in X$ is a solution to (1). Now, we may suppose that $d(x_1, x_0) \ne 0$. Let

$$\delta = d(x_1, x_0) > 0. \tag{30}$$

From (27), we have

$$d(x_{n+1}, x_n) \le \psi^n(\delta), \quad n = 0, 1, 2, \ldots. \tag{31}$$

Using the triangle inequality and (31), for all $m = 1, 2, 3, \ldots$, we have

$$\begin{aligned} d(x_n, x_{n+m}) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots \\ &\quad + d(x_{n+m-1}, x_{n+m}) \\ &\le \psi^n(\delta) + \psi^{n+1}(\delta) + \cdots + \psi^{n+m-1}(\delta) \\ &= \sum_{i=n}^{n+m-1} \psi^i(\delta) \le \sum_{i=n}^{\infty} \psi^i(\delta). \end{aligned} \tag{32}$$

On the other hand, since $\sum_{k=0}^{\infty} \psi^k(\delta) < \infty$, we have

$$\sum_{i=n}^{\infty} \psi^i(\delta) \longrightarrow 0 \quad \text{as } n \to \infty, \tag{33}$$

which implies that $\{x_n\} = \{T^n x_0\}$ is a Cauchy sequence in (X, d). Then there is some $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0. \tag{34}$$

On the other hand, from (23), we have

$$x_{2n} \in \operatorname{lev} S_{\preceq}, \quad n = 0, 1, 2, \ldots. \tag{35}$$

Since $S : X \to X$ is level closed from the left (from Assumption (A2)), passing to the limit as $n \to \infty$ and using (34), we obtain

$$x^* \in \operatorname{lev} S_{\preceq}, \tag{36}$$

that is,

$$x^* \preceq Sx^*. \tag{37}$$

Now, using (23), (37), and Assumption (A4), we obtain

$$d(Tx_{2n+1}, Tx^*) \le \psi(d(x_{2n+1}, x^*)), \quad n = 0, 1, 2, \ldots, \tag{38}$$

that is,

$$d(x_{2n+2}, Tx^*) \le \psi(d(x_{2n+1}, x^*)), \quad n = 0, 1, 2, \ldots. \tag{39}$$

Passing to the limit as $n \to \infty$, using (34), the continuity of ψ at 0, and the fact that $\psi(0) = 0$ (see Lemma 3), we get

$$d(x^*, Tx^*) = 0, \tag{40}$$

that is,

$$x^* = Tx^*. \tag{41}$$

Next, using (37), (41), and Assumption (A3), we obtain

$$x^* = Tx^* \succeq STx^* = Sx^*, \tag{42}$$

that is,

$$x^* \succeq Sx^*. \tag{43}$$

Since $\preceq$ is a partial order, inequalities (37) and (43) yield

$$x^* = Sx^*. \tag{44}$$

Further, (41) and (44) yield that $x^* \in X$ is a solution to problem (1). Therefore, (i) is proved.

Suppose now that $y^* \in X$ is another solution to (1) with $x^* \neq y^*$. Using Assumption (A4) and the result (i) in Lemma 3, we obtain

$$\begin{aligned} d(x^*, y^*) = d(Tx^*, Ty^*) &\le \psi(d(x^*, y^*)) \\ &< d(x^*, y^*), \end{aligned} \tag{45}$$

which is a contradiction. Therefore, $x^* \in X$ is the unique solution to (1), which proves (ii).

Passing to the limit as $m \to \infty$ in (32), we obtain estimate (16). In order to obtain estimate (17), observe that, by (26), we inductively obtain

$$\begin{gathered} d(x_{n+k}, x_{n+k+1}) \le \psi^{k+1}(d(x_{n-1}, x_n)), \\ n \ge 1,\ k \ge 0, \end{gathered} \tag{46}$$

and hence, similar to the derivation of (32), we obtain

$$d(x_{n+p}, x_n) \le \sum_{k=1}^{p} \psi^k(d(x_{n-1}, x_n)), \quad p \ge 0,\ n \ge 1. \tag{47}$$

Now, passing to the limit as $p \to \infty$, (17) follows.

The proof is complete. □

Observe that Theorem 4 holds true if we replace Assumption (A2) by the following.

Assumption (A2)′. The operator $S : X \to X$ is level closed from the right; that is, the set

$$\operatorname{lev} S_{\succeq} = \{x \in X : x \succeq Sx\} \tag{48}$$

is nonempty and closed.

As a consequence, we have the following result.

Theorem 5. *Suppose that Assumptions (A1) and (A2)′–(A4) are satisfied. Then*

(i) *for any $x_0 \in \operatorname{lev} S_{\succeq}$, the Picard sequence $\{T^n x_0\}$ converges to some $x^* \in X$, which is a solution to (1),*

(ii) *$x^* \in X$ is the unique solution to (1),*

(iii) *the following estimates*

$$\begin{gathered} d(T^n x_0, x^*) \le \mu_n(d(Tx_0, x_0)), \quad n = 0, 1, 2, \ldots, \\ d(T^n x_0, x^*) \le \mu_1\left(d\left(T^{n-1} x_0, T^n x_0\right)\right), \\ n = 1, 2, 3, \ldots \end{gathered} \tag{49}$$

hold.

Taking $S = I_X$ (the identity operator), we obtain immediately from Theorem 4 (or from Theorem 5) the following fixed point result.

Corollary 6. *Let (X, d) be a complete metric space. Let $T : X \to X$ be a given mapping. Suppose that there exists some $\psi \in \Psi$ such that*

$$d(Tx, Ty) \le \psi(d(x, y)), \quad (x, y) \in X \times X. \tag{50}$$

Then

(i) *for any $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to some $x^* \in X$, which is a fixed point of T,*

(ii) *$x^* \in X$ is the unique fixed point of T,*

(iii) *the following estimates*

$$\begin{gathered} d(T^n x_0, x^*) \le \mu_n(d(Tx_0, x_0)), \quad n = 0, 1, 2, \ldots, \\ d(T^n x_0, x^*) \le \mu_1\left(d\left(T^{n-1} x_0, T^n x_0\right)\right), \\ n = 1, 2, 3, \ldots \end{gathered} \tag{51}$$

hold.

Remark 7. Observe that all the obtained results hold true if we replace the partial order $\preceq$ by any binary relation $\mathscr{R}$ which is antisymmetric; that is, $\mathscr{R}$ satisfies

$$\begin{gathered} (x, y) \in X \times X, \\ x \mathscr{R} y,\ y \mathscr{R} x \Longrightarrow x = y. \end{gathered} \tag{52}$$

We end the paper with the following illustrative example.

Example 8. Let $X = [0, \infty)$ and d be the metric on X defined by

$$d(x, y) = |x - y|, \quad (x, y) \in X \times X. \tag{53}$$

Then (X, d) is a complete metric space. Let $\mathscr{R}$ be the binary relation on X defined by

$$\mathscr{R} = \{(x, x) : x \in X\} \cup \{(0, 2)\}. \tag{54}$$

Consider the partial order on X defined by

$$(x, y) \in X \times X, \quad x \preceq y \Longleftrightarrow (x, y) \in \mathscr{R}. \tag{55}$$

Let us define the pair of mappings $T, S : X \to X$ by

$$Tx = \begin{cases} x & \text{if } x \notin \{0, 2\}, \\ 2 & \text{otherwise}, \end{cases}$$
$$Sx = \begin{cases} 2 & \text{if } x \in [0, 2], \\ 1 & \text{if } x > 2. \end{cases} \tag{56}$$

Observe that, in this case, we have

$$\operatorname{lev} S_{\preceq} = \{x \in X : x \preceq Sx\} = \{0, 2\}, \tag{57}$$

which is nonempty and closed set. Therefore, the operator $S : X \to X$ is level closed from the left, and Assumption (A2) is satisfied. Moreover, we have

$$\{x \in X : Sx \preceq x\} = \{2\}. \tag{58}$$

In order to check the validity of Assumption (A3), let $x \in X$ be such that $x \preceq Sx$; that is, $x \in \{0, 2\}$. If $x = 0$, then $Tx = T0 = 2$ and $STx = ST0 = S2 = 2$. Then $STx \preceq Tx$. If $x = 2$, then $Tx = T2 = 2$ and $STx = ST2 = S2 = 2$. Then $STx \preceq Tx$. Now, let $x \in X$ be such that $Sx \preceq x$; that is, $x = 2$. In this case, we have $STx = ST2 = S2 = 2$ and $Tx = T2 = 2$. Then $Tx \preceq STx$. Therefore, Assumption (A3) is satisfied. Now, let $(x, y) \in X \times X$ be such that $x \preceq Sx$ and $Sy \preceq y$; that is, $x \in \{0, 2\}$ and $y = 2$. For $(x, y) = (0, 2)$, we have

$$d(Tx, Ty) = d(T0, T2) = d(2, 2) = 0 \le \psi(d(0, 2)), \tag{59}$$

for every $\psi \in \Psi$. For $(x, y) = (2, 2)$, we have

$$d(Tx, Ty) = d(T2, T2) = 0 \le \psi(d(2, 2)) = \psi(0), \tag{60}$$

for every $\psi \in \Psi$. Therefore, Assumption (A4) is satisfied. Now, applying Theorem 4, we deduce that problem (1) has a unique solution $x^* \in X$. Clearly, in our case, we have $x^* = 2$.

Remark 9. Note that Theorem 4 (or Theorem 5) provides us just the existence and uniqueness of a common fixed point of the operators $T, S : X \to X$. However, the uniqueness of the fixed points of T is not satisfied in general. As we observe in Example 8, the operator T has infinitely many fixed points.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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