Journal of
Mathematical
Inequalities
Volume 12, Number 3 (2018), 827–851 doi:10.7153/jmi-2018-12-62

GENERALIZED FRACTIONAL MAXIMAL FUNCTIONS IN LORENTZ SPACES Λ

R. CH. MUSTAFAYEV AND N. BILGIÇLI

(*Communicated by J. Pečarić*)

Abstract. In this paper we give the complete characterization of the boundedness of generalized fractional maximal operator

$$M_{\phi,\Lambda^{\alpha}(b)}f(x) := \sup_{Q\ni x} \frac{\|f\chi_Q\|_{\Lambda^{\alpha}(b)}}{\phi(|Q|)} \qquad (x\in\mathbb{R}^n),$$

between the classical Lorentz spaces $\Lambda^p(v)$ and $\Lambda^q(w)$, as well as between $\Lambda^p(v)$ and weak-type Lorentz spaces $\Lambda^{q,\infty}(w)$, and between $\Lambda^{p,\infty}(v)$ and $\Lambda^{q,\infty}(w)$, and between $\Lambda^{p,\infty}(v)$ and $\Lambda^q(w)$, for appropriate functions ϕ, where $0<p,q,\alpha<\infty$, v,w,b are weights on $(0,\infty)$ such that $0<B(t):=\int_0^t b<\infty$, $t>0$, $B\in\Delta_2$ and $B(t)/t^r$ is quasi-increasing for some $0<r\leqslant 1$.

1. Introduction

Throughout the paper, we always denote by c or C a positive constant, which is independent of main parameters but it may vary from line to line. However a constant with subscript such as c_1 does not change in different occurrences. By $a\lesssim b$, we mean that $a\leqslant \lambda b$, where $\lambda>0$ depends on inessential parameters. If $a\lesssim b$ and $b\lesssim a$, we write $a\approx b$ and say that a and b are equivalent. By a cube, we mean an open cube with sides parallel to the coordinate axes.

Let Ω be any measurable subset of $\mathbb{R}^n$, $n\geqslant 1$. Let $\mathfrak{M}(\Omega)$ denote the set of all measurable functions on Ω and $\mathfrak{M}_0(\Omega)$ the class of functions in $\mathfrak{M}(\Omega)$ that are finite a.e. The symbol $\mathfrak{M}^+(\Omega)$ stands for the collection of all $f\in\mathfrak{M}(\Omega)$ which are non-negative on Ω. The symbol $\mathfrak{M}^+((0,\infty);\downarrow)$ is used to denote the subset of those functions from $\mathfrak{M}^+(0,\infty)$ which are non-increasing on $(0,\infty)$. Denote by $\mathfrak{M}^{\mathrm{rad},\downarrow}\equiv\mathfrak{M}^{\mathrm{rad},\downarrow}(\mathbb{R}^n)$ the set of all measurable, non-negative, radially decreasing functions on $\mathbb{R}^n$, that is,

$$\mathfrak{M}^{\mathrm{rad},\downarrow} := \{f\in\mathfrak{M}(\mathbb{R}^n) : f(x)=h(|x|),\, x\in\mathbb{R}^n \text{ with } h\in\mathfrak{M}^+((0,\infty);\downarrow)\}.$$

The family of all weight functions (also called just weights) on Ω, that is, locally integrable non-negative functions on Ω, is given by $\mathscr{W}(\Omega)$. Everywhere in the paper, u, v and w are weights.

Mathematics subject classification (2010): 42B25, 42B35.
Keywords and phrases: Maximal functions, classical and weak-type Lorentz spaces, iterated Hardy inequalities involving suprema, weights.

For $p \in (0,\infty]$ and $w \in \mathfrak{M}^+(\Omega)$, we define the functional $\|\cdot\|_{p,w,\Omega}$ on $\mathfrak{M}(\Omega)$ by

$$\|f\|_{p,w,\Omega} := \begin{cases} (\int_\Omega |f(x)|^p w(x)\,dx)^{1/p} & \text{if} \quad p < \infty, \\ \operatorname{ess\,sup}_\Omega |f(x)| w(x) & \text{if} \quad p = \infty. \end{cases}$$

If, in addition, $w \in \mathscr{W}(\Omega)$, then the weighted Lebesgue space $L^p(w,\Omega)$ is given by

$$L^p(w,\Omega) = \{f \in \mathfrak{M}(\Omega) : \ \|f\|_{p,w,\Omega} < \infty\}$$

and it is equipped with the quasi-norm $\|\cdot\|_{p,w,\Omega}$.

When $w \equiv 1$ on Ω, we write simply $L^p(\Omega)$ and $\|\cdot\|_{p,\Omega}$ instead of $L^p(w,\Omega)$ and $\|\cdot\|_{p,w,\Omega}$, respectively.

Denote by

$$V(x) := \int_0^x v(t)\,dt \text{ and } W(x) := \int_0^x w(t)\,dt \text{ for all } x > 0.$$

A quasi-Banach space X is a complete metrizable real vector space whose topology is given by a quasi-norm $\|\cdot\|$ satisfying the following three conditions: $\|x\| > 0$, $x \in X$, $x \neq 0$; $\|\lambda x\| = |\lambda| \|x\|$, $\lambda \in \mathbb{R}$, $x \in X$; and $\|x_1 + x_2\| \leqslant C(\|x_1\| + \|x_2\|)$, $x_1, x_2 \in X$, where $C \geqslant 1$ is a constant independent of x_1 and x_2.

A quasi-Banach function space on a measure space $(\mathbb{R}^n, dx)$ is defined to be a quasi-Banach space X which is a subspace of $\mathfrak{M}_0(\mathbb{R}^n)$ (the topological linear space of all equivalence classes of the real Lebesgue measurable functions equipped with the topology of convergence in measure) such that there exists $h \in X$ with $h > 0$ a.e. and if $|f| \leqslant |g|$ a.e., where $g \in X$ and $f \in \mathfrak{M}_0(\mathbb{R}^n)$, then $f \in X$ and $\|f\|_X \leqslant \|g\|_X$.

A quasi-Banach function space X is said to satisfy a lower r-estimate, $0 < r < \infty$, if there exists a constant C such that the inequality

$$\left(\sum_{i=1}^n \|f_i\|_X^r \right)^{1/r} \leqslant C \left\| \sum_{i=1}^n f_i \right\|_X$$

holds for every finite set of functions $\{f_1, \ldots, f_n\} \subset X$ with pairwise disjoint supports (see [37, 1.f.4]).

Suppose f is a measurable a.e. finite function on $\mathbb{R}^n$. Then its non-increasing rearrangement f^* is given by

$$f^*(t) = \inf\{\lambda > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| \leqslant t\}, \quad t \in (0,\infty),$$

and let f^{**} denotes the Hardy-Littlewood maximal function of f^*, i.e.

$$f^{**}(t) := \frac{1}{t} \int_0^t f^*(\tau)\,d\tau, \quad t > 0.$$

A quasi-Banach function space $(X, \|\cdot\|_X)$ of real-valued, locally integrable, Lebesgue measurable functions on $\mathbb{R}^n$ is said to be a rearrangement-invariant (r.i.) space if it satisfies the following conditions:

1. If $g^* \leqslant f^*$ and $f \in X$, then $g \in X$ with $\|g\|_X \leqslant \|f\|_X$.
2. If A is a Lebesgue measurable set of finite measure, then $\chi_A \in X$.
3. $0 \leqslant f_n \uparrow$, $\sup_{n\in\mathbb{N}} \|f_n\|_X \leqslant M$ imply that $f = \sup_{n\in\mathbb{N}} f_n \in X$ and $\|f\|_X = \sup_{n\in\mathbb{N}} \|f_n\|_X$

(see, for instance, [2]).

For each r.i. space X on $\mathbb{R}^n$, a r.i. space $\bar{X}$ on $(0,+\infty)$ is associated such that $f \in X$ if and only if $f^* \in \bar{X}$ and $\|f\|_X = \|f^*\|_{\bar{X}}$ (see [3]).

Quite many familiar function spaces can be defined using the non-increasing rearrangement of a function. One of the most important classes of such spaces are the so-called classical Lorentz spaces.

Let $p \in (0,\infty)$ and $w \in \mathscr{W}(0,\infty)$. The classical Lorentz spaces $\Lambda^p(w)$ consist of all functions $f \in \mathfrak{M}(\mathbb{R}^n)$ for which $\|f\|_{\Lambda^p(w)} := \|f^*\|_{p,w,(0,\infty)} < \infty$. For more information about the Lorentz Λ spaces see e.g. [9] and the references therein.

A weak-type modification of the space $\Lambda^p(w)$ is defined by (cf. [12, 49])

$$\Lambda^{p,\infty}(w) := \left\{ f \in \mathfrak{M}(\mathbb{R}^n) : \|f\|_{\Lambda^{p,\infty}(w)} := \sup_{t>0} f^*(t) W(t)^{1/p} < \infty \right\}.$$

Recall that classical and weak-type Lorentz spaces include many familiar spaces (see, for instance, [18]).

A function $\phi : (0,\infty) \to (0,\infty)$ is said to satisfy the Δ_2-condition, denoted $\phi \in \Delta_2$, if for some $C > 0$

$$\phi(2t) \leqslant C\phi(t) \quad \text{for all} \quad t > 0.$$

Suppose $0 < p < \infty$ and let w be a weight on $(0,\infty)$ such that $W \in \Delta_2$ and $W(\infty) = \infty$. Then the classical Lorentz space $\Lambda^p(w)$ is a r.i. quasi-Banach function space (see, for instance, [10, Section 2.2] and [31]).

THEOREM 1.1. [31, Theorem 7] *Let w be a weight function such that $W \in \Delta_2$. Given $0 < p, r < \infty$, the following assertions are equivalent:*
(i) *$\Lambda^p(w)$ satisfies a lower r-estimate.*
(ii) *$W(t)/t^{p/r}$ is quasi-increasing and $r \geqslant p$.*

The study of maximal operators is one of the most important topics in harmonic analysis. These significant non-linear operators, whose behavior are very informative in particular in differentiation theory, provided the understanding and the inspiration for the development of the general class of singular and potential operators (see, for instance, [51, 30, 21, 54, 52, 28, 29]).

Suppose that X is a quasi-Banach space of measurable functions defined on $\mathbb{R}^n$. Given a function $\phi : (0,\infty) \to (0,\infty)$, denote for every $f \in X_{\mathrm{loc}} := \{ f \in \mathfrak{M}_0(\mathbb{R}^n) : f\chi_Q \in X$ for every cube $Q \subset \mathbb{R}^n \}$ by

$$M_{\phi,X} f(x) := \sup_{Q \ni x} \frac{\|f\chi_Q\|_X}{\phi(|Q|)} \qquad (x \in \mathbb{R}^n). \tag{1.1}$$

It is easy to see that $M_{\phi,X}f$ is a lower-semicontinuous function.

A function $\phi:(0,\infty)\to(0,\infty)$ is said to be quasi-increasing (quasi-decreasing), if for some $C>0$

$$\phi(t_1)\leqslant C\phi(t_2) \qquad (\phi(t_2)\leqslant c\phi(t_1))$$

holds whenever $0<t_1\leqslant t_2<\infty$.

A function $\phi:(0,\infty)\to(0,\infty)$ is said to satisfy the Q_r-condition, $0<r<\infty$, denoted $\phi\in Q_r$, if for some constant $C>0$

$$\phi\left(\sum_{i=1}^n t_i\right)\leqslant C\left(\sum_{i=1}^n \phi(t_i)^r\right)^{1/r}$$

holds for every finite set of non-negative real numbers $\{t_1,\ldots,t_n\}$.

In this paper we study the boundedness of $M_{\phi,X}$ between the classical Lorentz spaces $\Lambda^p(v)$ and $\Lambda^q(w)$, as well as between $\Lambda^p(v)$ and weak-type Lorentz spaces $\Lambda^{q,\infty}(w)$, and between $\Lambda^{p,\infty}(v)$ and $\Lambda^{q,\infty}(w)$, and between $\Lambda^{p,\infty}(v)$ and $\Lambda^q(w)$

Our main result reads as follows.

THEOREM 1.2. *Let $0<p,q<\infty$, $0<r<\infty$. Assume that $\phi\in Q_r$ is a quasi-increasing function on $(0,\infty)$. Suppose that X is a r.i. quasi-Banach function space satisfying a lower r-estimate. Then:*

(a) *$M_{\phi,X}$ is bounded from $\Lambda^p(v)$ to $\Lambda^q(w)$, that is, the inequality*

$$\|M_{\phi,X}f\|_{\Lambda^q(w)}\leqslant C\|f\|_{\Lambda^p(v)}$$

holds for all $f\in\mathfrak{M}(\mathbb{R}^n)$ if and only if the inequality

$$\left(\int_0^\infty\left[\sup_{\tau>t}\frac{\|\psi\chi_{[0,\tau)}\|_{\bar{X}}}{\phi(\tau)}\right]^q w(t)\,dt\right)^{1/q}\leqslant C\left(\int_0^\infty \psi(t)^p v(t)\,dt\right)^{1/p}$$

holds for all $\psi\in\mathfrak{M}^+((0,\infty);\downarrow)$.

(b) *$M_{\phi,X}$ is bounded from $\Lambda^p(v)$ to $\Lambda^{q,\infty}(w)$, that is, the inequality*

$$\|M_{\phi,X}f\|_{\Lambda^{q,\infty}(w)}\leqslant C\|f\|_{\Lambda^p(v)}$$

holds for all $f\in\mathfrak{M}(\mathbb{R}^n)$ if and only if the inequality

$$\sup_{t>0}W(t)^{1/q}\sup_{\tau>t}\frac{\|\psi\chi_{[0,\tau)}\|_{\bar{X}}}{\phi(\tau)}\leqslant C\left(\int_0^\infty \psi(t)^p v(t)\,dt\right)^{1/p}$$

holds for all $\psi\in\mathfrak{M}^+((0,\infty);\downarrow)$.

(c) *$M_{\phi,X}$ is bounded from $\Lambda^{p,\infty}(v)$ to $\Lambda^{q,\infty}(w)$, that is, the inequality*

$$\|M_{\phi,X}f\|_{\Lambda^{q,\infty}(w)}\leqslant C\|f\|_{\Lambda^{p,\infty}(v)}$$

holds for all $f\in\mathfrak{M}(\mathbb{R}^n)$ if and only if the inequality

$$\sup_{t>0}W(t)^{1/q}\sup_{\tau>t}\frac{\|\psi\chi_{[0,\tau)}\|_{\bar{X}}}{\phi(\tau)}\leqslant C\sup_{t>0}V(t)^{1/q}\psi(t)$$

holds for all $\psi \in \mathfrak{M}^+((0,\infty);\downarrow)$.

(d) $M_{\phi,X}$ *is bounded from* $\Lambda^{p,\infty}(v)$ *to* $\Lambda^q(w)$, *that is, the inequality*

$$\|M_{\phi,X} f\|_{\Lambda^q(w)} \leqslant C\|f\|_{\Lambda^{p,\infty}(v)}$$

holds for all $f \in \mathfrak{M}(\mathbb{R}^n)$ *if and only if the inequality*

$$\left(\int_0^\infty \left[\sup_{\tau>t} \frac{\|\psi\chi_{[0,\tau)}\|_{\bar{X}}}{\phi(\tau)}\right]^q w(t)\,dt\right)^{1/q} \leqslant C \sup_{t>0} V(t)^{1/q}\psi(t)$$

holds for all $\psi \in \mathfrak{M}^+((0,\infty);\downarrow)$.

Let $u \in \mathscr{W}(0,\infty) \cap C(0,\infty)$, $b \in \mathscr{W}(0,\infty)$ and $B(t) := \int_0^t b(s)\,ds$. Assume that b is such that $0 < B(t) < \infty$ for every $t \in (0,\infty)$. The iterated Hardy-type operator involving suprema $T_{u,b}$ is defined at $g \in \mathfrak{M}^+(0,\infty)$ by

$$(T_{u,b}g)(t) := \sup_{t\leqslant\tau<\infty} \frac{u(\tau)}{B(\tau)} \int_0^\tau g(y)b(y)\,dy, \qquad t \in (0,\infty).$$

Such operators have been found indispensable in the search for optimal pairs of rearrangement-invariant norms for which a Sobolev-type inequality holds (cf. [32]). They constitute a very useful tool for characterization of the associate norm of an operator-induced norm, which naturally appears as an optimal domain norm in a Sobolev embedding (cf. [45], [46]). Supremum operators are also very useful in limiting interpolation theory as can be seen from their appearance for example in [20, 17, 16, 47].

In the present paper we also give solution of the inequality

$$\|T_{u,b}f\|_{q,w,(0,\infty)} \leqslant c\|f\|_{p,v,(0,\infty)}, \qquad f \in \mathfrak{M}^{\downarrow}(0,\infty) \tag{1.2}$$

when $p = \infty$ or $q = \infty$ (see Theorems 2.6, 2.7 and 2.8). Recall that the complete characterization of inequality (1.2) for $0 < q < \infty$, $0 < p < \infty$ is given in [24] (see Theorem 2.3).

In particular case, when $X = \Lambda^\alpha(b)$, $0 < \alpha < \infty$, we are able to give the complete characterization of the boundedness of $M_{\phi,X}$ between the classical Lorentz spaces Λ (see Theorem 3.14, 3.16, 3.18 and 3.20). We reduce the problem to the boundedness of the operator $T_{u,b}$ in weighted Lebesgue spaces on the cone of non-negative non-increasing functions (see, Theorem 1.2 applied with $X = \Lambda^\alpha(b)$, as well as Theorems 3.13, 3.15, 3.17 and 3.19).

EXAMPLE 1.3. The main example is the Hardy-Littlewood maximal function which is defined for locally integrable functions f on $\mathbb{R}^n$ by

$$Mf(x) := \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f(y)|\,dy, \ \ x \in \mathbb{R}^n.$$

Obviously, $M_{\phi,\Lambda^\alpha(b)} = M$, when $\alpha = 1$, $b \equiv 1$ and $\phi(t) = t$ $(t > 0)$.

The first results on the problem of boundedness of the Hardy-Littlewood maximal operator between the classical Lorentz spaces $\Lambda^p(v)$ and $\Lambda^q(w)$ were obtained by Boyd

[5] and in an explicit form by Ariño and Muckenhoupt [1] when $p=q$ and $w=v$. The problem with $w \neq v$ and $p \neq q$, $1<p,q<\infty$ was first successfully solved by Sawyer [48] using duality argument. Stepanov [53] applied a different approach which enabled him to extend the range of parameters to $0<q<\infty$, $1<p<\infty$. He also proved the appropriate analogue of Sawyer's duality principle in the case $0<p<1$, and extended the range of parameters to $0<p<1<q<\infty$. The case $0<p\leqslant q\leqslant 1$ have been obtained (in different ways) by several authors (see [11, 12]). The missing case $0<q<p<1$ was considered in [26] and [7]. Full characterizations for all range of parameters using different discretization techniques were obtained in the papers [4] and [22]. Many articles on this topic followed, providing the results for a wider range of parameters (see for instance survey [9], the monographs [33, 34], for the latest development of this subject see [27, 23], and references given there). The boundedness of M from $\Lambda^p(v)$ to $\Lambda^{q,\infty}(w)$ was characterized in [6, 13, 9]. Necessary and sufficient conditions for the booundedness of M from $\Lambda^{p,\infty}(v)$ to $\Lambda^{q,\infty}(w)$ were established in [49].

EXAMPLE 1.4. The fractional maximal operator, M_γ, $\gamma\in(0,n)$, is defined at $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$(M_\gamma f)(x) := \sup_{Q\ni x} |Q|^{\gamma/n-1}\int_Q |f(y)|\,dy, \quad x\in\mathbb{R}^n.$$

Note that $M_{\phi,\Lambda^\alpha(b)} = M_\gamma$, when $\alpha=1$, $b\equiv 1$ and $\phi(t)=t^{1-\gamma/n}$ $(t>0)$ with $0<\gamma<n$.

The characterization of the boundedness of M_γ between $\Lambda^p(v)$ and $\Lambda^q(w)$ was obtained in [15] for the particular case when $1<p\leqslant q<\infty$ and in [40, Theorem 2.10] in the case of more general operators and for extended range of p and q.

EXAMPLE 1.5. Let $s\in(0,\infty)$, $\gamma\in[0,n)$ and $\mathbb{A}=(A_0,A_\infty)\in\mathbb{R}^2$. Denote by

$$\ell^{\mathbb{A}}(t) := (1+|\log t|)^{A_0}\chi_{[0,1]}(t) + (1+|\log t|)^{A_\infty}\chi_{[1,\infty)}(t), \quad (t>0).$$

Recall that the fractional maximal operator $M_{s,\gamma,\mathbb{A}}$ at $f\in\mathfrak{M}(\mathbb{R}^n)$ defined in [18] by

$$(M_{s,\gamma,\mathbb{A}} f)(x) := \sup_{Q\ni x}\frac{\|f\chi_Q\|_s}{\|\chi_Q\|_{sn/(n-\gamma),\mathbb{A}}}, \quad x\in\mathbb{R}^n$$

satisfies the following equivalency

$$(M_{s,\gamma,\mathbb{A}} f)(x) \approx \sup_{Q\ni x}\frac{\|f\chi_Q\|_s}{|Q|^{(n-\gamma)/(sn)}\ell^{\mathbb{A}}(|Q|)}, \quad x\in\mathbb{R}^n.$$

Hence, if $s=1$, $\gamma=0$ and $\mathbb{A}=(0,0)$, then $M_{s,\gamma,\mathbb{A}}$ is equivalent to the classical Hardy-Littlewood maximal operator M. If $s=1$, $\gamma\in(0,n)$ and $\mathbb{A}=(0,0)$, then $M_{s,\gamma,\mathbb{A}}$ is equivalent to the usual fractional maximal operator M_γ. Moreover, if $s=1$, $\gamma\in[0,n)$ and $\mathbb{A}\in\mathbb{R}^2$, then $M_{s,\gamma,\mathbb{A}}$ is the fractional maximal operator which corresponds to potentials with logarithmic smoothness treated in [42, 43]. In particular, if $\gamma=0$, then $M_{1,\gamma,\mathbb{A}}$ is the maximal operator of purely logarithmic order.

Note that $M_{\phi,\Lambda^\alpha(b)} \approx M_{s,\gamma,\mathbb{A}}$, when $\alpha = s$, $b \equiv 1$ and $\phi(t) = t^{(n-\gamma)/(sn)} \ell^{\mathbb{A}}(t)$, $(t > 0)$ with $0 < \gamma < n$ and $\mathbb{A} = (A_0, A_\infty) \in \mathbb{R}^2$.

The complete characterization of the boundedness of $M_{s,\gamma,\mathbb{A}}$ between $\Lambda^p(v)$ and $\Lambda^q(w)$, as well as between $\Lambda^p(v)$ and $\Lambda^{q,\infty}(w)$, and between $\Lambda^{p,\infty}(v)$ and $\Lambda^{q,\infty}(w)$, and between $\Lambda^{p,\infty}(v)$ and $\Lambda^q(w)$ was given in [18, p. 17 and p. 34]. Full proofs and some further extensions and applications can be found in [18, 19].

EXAMPLE 1.6. Given p and q, $0 < p, q < \infty$, let $M_{p,q}$ denote the maximal operator associated to the Lorentz $L^{p,q}$ spaces defined by

$$M_{p,q} f(x) := \sup_{Q \ni x} \frac{\|f\chi_Q\|_{p,q}}{\|\chi_Q\|_{p,q}} = \sup_{Q \ni x} \frac{\|f\chi_Q\|_{p,q}}{|Q|^{1/p}},$$

where $\|\cdot\|_{p,q}$ is the usual Lorentz norm

$$\|f\|_{p,q} := \left(\int_0^\infty \left[\tau^{1/p} f^*(\tau)\right]^q \frac{d\tau}{\tau} \right)^{1/q}.$$

This operator was introduced by Stein in [50] in order to obtain certain endpoint results in differentiation theory. The operator $M_{p,q}$ have been also considered by other authors, for instance see [39, 35, 2, 44, 36]. The boundedness of $M_{p,q}$ between $\Lambda^p(v)$ and $\Lambda^q(w)$ was studied in [8].

Evidently, $M_{\phi,\Lambda^\alpha(b)} = M_{p,q}$, when $\alpha = q$, $b(t) = t^{q/p-1}$ and $\phi(t) = t^{1/p}$ $(t > 0)$.

The paper is organized as follows. In Section 2, for the convenience of the reader, we recall the above-mentioned characterization of inequality (1.2), when $0 < p, q < \infty$, and give solution of this inequality, when $p = \infty$ or $q = \infty$. The main results are proved in Section 3.

2. Restricted inequalities for $T_{u,b}$

In this section, we recall the characterization of (1.2), when $0 < p, q < \infty$, and give solution of this inequality, when $p = \infty$ or $q = \infty$.

REMARK 2.1. Inequality (1.2) was characterized in [25, Theorem 3.5] under additional condition

$$\sup_{0<t<\infty} \frac{u(t)}{B(t)} \int_0^t \frac{b(\tau)}{u(\tau)} d\tau < \infty.$$

Note that the case when $0 < p \leqslant 1 < q < \infty$ was not considered in [25]. It is also worth to mention that in the case when $1 < p < \infty$, $0 < q < p < \infty$, $q \neq 1$ [25, Theorem 3.5] contains only discrete condition. In [26] the new reduction theorem was obtained when $0 < p \leqslant 1$, and this technique allowed to characterize inequality (1.2) when $b \equiv 1$, and in the case when $0 < q < p \leqslant 1$, [26] contains only discrete condition.

We adopt the following conventions:

CONVENTION 2.2. (i) Throughout the paper we put $0\cdot\infty=0$, $\infty/\infty=0$ and $0/0=0$.

(ii) If $p\in[1,+\infty]$, we define p' by $1/p+1/p'=1$.

(iii) If $0<q<p<\infty$, we define r by $1/r=1/q-1/p$.

THEOREM 2.3. [24, Theorems 5.1 and 5.5] *Let $0<p,q<\infty$ and let $u\in\mathscr{W}(0,\infty)\cap C(0,\infty)$. Assume that $b,v,w\in\mathscr{W}(0,\infty)$ is such that $0<B(x)<\infty$, $0<V(x)<\infty$ and $0<W(x)<\infty$ for all $x>0$. Then inequality (1.2) is satisfied with the best constant C if and only if the following holds:*

(i) *$1<p\leqslant q$ and $A_1+A_2<\infty$, where*

$$A_1:=\sup_{x>0}\left(\left[\sup_{x\leqslant\tau<\infty}\frac{u(\tau)}{B(\tau)}\right]^q W(x)+\int_x^\infty\left[\sup_{t\leqslant\tau<\infty}\frac{u(\tau)}{B(\tau)}\right]^q w(t)\,dt\right)^{1/q}\times\left(\int_0^x\left(\frac{B(y)}{V(y)}\right)^{p'}v(y)\,dy\right)^{1/p'},$$

$$A_2:=\sup_{x>0}\left(\left[\sup_{x\leqslant\tau<\infty}\frac{u(\tau)}{V^2(\tau)}\right]^q W(x)+\int_x^\infty\left[\sup_{t\leqslant\tau<\infty}\frac{u(\tau)}{V^2(\tau)}\right]^q w(t)\,dt\right)^{1/q}\left(\int_0^x V^{p'}(y)v(y)\,dy\right)^{1/p'},$$

and in this case $C\approx A_1+A_2$;

(ii) *$1=p\leqslant q$ and $B_1+B_2<\infty$, where*

$$B_1:=\sup_{x>0}\left(\left[\sup_{x\leqslant\tau<\infty}\frac{u(\tau)}{B(\tau)}\right]^q W(x)+\int_x^\infty\left[\sup_{t\leqslant\tau<\infty}\frac{u(\tau)}{B(\tau)}\right]^q w(t)\,dt\right)^{1/q}\left(\sup_{0<y\leqslant x}\frac{B(y)}{V(y)}\right),$$

$$B_2:=\sup_{x>0}\left(\left[\sup_{x\leqslant\tau<\infty}\frac{u(\tau)}{V^2(\tau)}\right]^q W(x)+\int_x^\infty\left[\sup_{t\leqslant\tau<\infty}\frac{u(\tau)}{V^2(\tau)}\right]^q w(t)\,dt\right)^{1/q}V(x),$$

and in this case $C\approx B_1+B_2$;

(iii) *$\max\{q,1\}<p$ and $C_1+C_2+C_3+C_4<\infty$, where*

$$C_1:=\left(\int_0^\infty\left(\int_x^\infty\left[\sup_{t\leqslant\tau<\infty}\frac{u(\tau)}{B(\tau)}\right]^q w(t)\,dt\right)^{r/p}\left[\sup_{x\leqslant\tau<\infty}\frac{u(\tau)}{B(\tau)}\right]^q\times\left(\int_0^x\left(\frac{B(y)}{V(y)}\right)^{p'}v(y)\,dy\right)^{r/p'}w(x)\,dx\right)^{1/r},$$

$$C_2:=\left(\int_0^\infty W^{r/p}(x)\left[\sup_{x\leqslant\tau<\infty}\left[\sup_{\tau\leqslant y<\infty}\frac{u(y)}{B(y)}\right]\left(\int_0^\tau\left(\frac{B(y)}{V(y)}\right)^{p'}v(y)\,dy\right)^{1/p'}\right]^r w(x)\,dx\right)^{1/r},$$

$$C_3:=\left(\int_0^\infty\left(\int_x^\infty\left[\sup_{t\leqslant\tau<\infty}\frac{u(\tau)}{V^2(\tau)}\right]^q w(t)\,dt\right)^{r/p}\left[\sup_{x\leqslant\tau<\infty}\frac{u(\tau)}{V^2(\tau)}\right]^q\times\left(\int_0^x V^{p'}(y)v(y)\,dy\right)^{r/p'}w(x)\,dx\right)^{1/r},$$

$$C_4 := \left(\int_0^\infty W^{r/p}(x) \left[\sup_{x \leqslant \tau < \infty} \left[\sup_{\tau \leqslant y < \infty} \frac{u(y)}{V^2(y)} \right] \left(\int_0^\tau V^{p'}(y) v(y)\, dy \right)^{1/p'} \right]^r w(x)\, dx \right)^{1/r},$$

and in this case $C \approx C_1 + C_2 + C_3 + C_4$;

(iv) $q < 1 = p$ *and* $D_1 + D_2 + D_3 + D_4 < \infty$, *where*

$$D_1 := \left(\int_0^\infty \left(\int_x^\infty \left[\sup_{t \leqslant \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q w(t)\, dt \right)^{r/p} \left[\sup_{x \leqslant \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q \right.$$
$$\left. \times \left(\sup_{0 < y \leqslant x} \frac{B(y)}{V(y)} \right)^r w(x)\, dx \right)^{1/r},$$

$$D_2 := \left(\int_0^\infty W^{r/p}(x) \left[\sup_{x \leqslant \tau < \infty} \left[\sup_{\tau \leqslant y < \infty} \frac{u(y)}{B(y)} \right] \left(\sup_{0 < y \leqslant \tau} \frac{B(y)}{V(y)} \right) \right]^r w(x)\, dx \right)^{1/r},$$

$$D_3 := \left(\int_0^\infty \left(\int_x^\infty \left[\sup_{t \leqslant \tau < \infty} \frac{u(\tau)}{V^2(\tau)} \right]^q w(t)\, dt \right)^{r/p} \left[\sup_{x \leqslant \tau < \infty} \frac{u(\tau)}{V^2(\tau)} \right]^q V^r(x) w(x)\, dx \right)^{1/r},$$

$$D_4 := \left(\int_0^\infty W^{r/p}(x) \left[\sup_{x \leqslant \tau < \infty} \left[\sup_{\tau \leqslant y < \infty} \frac{u(y)}{V^2(y)} \right] V(\tau) \right]^r w(x)\, dx \right)^{1/r},$$

and in this case $C \approx D_1 + D_2 + D_3 + D_4$;

(v) $p \leqslant \min\{q, 1\}$ *and* $E_1 + E_2 < \infty$, *where*

$$E_1 := \sup_{x > 0} \left(\left[\sup_{x \leqslant \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q W(x) + \int_x^\infty \left[\sup_{t \leqslant \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q w(t)\, dt \right)^{1/q} \sup_{0 < y \leqslant x} \frac{B(y)}{V^{1/p}(y)},$$

$$E_2 := \sup_{x > 0} \left(\left[\sup_{x \leqslant y < \infty} \frac{u^p(y)}{V^2(y)} \right]^{q/p} W(x) + \int_x^\infty \left[\sup_{t \leqslant y < \infty} \frac{u^p(y)}{V^2(y)} \right]^{q/p} w(t)\, dt \right)^{1/q} V^{1/p}(x),$$

and in this case $C \approx E_1 + E_2$;

(vi) $q < p \leqslant 1$ *and* $F_1 + F_2 + F_3 + F_4 < \infty$, *where*

$$F_1 := \left(\int_0^\infty W^{r/p}(x) \left[\sup_{x \leqslant \tau < \infty} \left[\sup_{\tau \leqslant y < \infty} \frac{u(y)}{B(y)} \right]^p \left(\sup_{0 < y \leqslant \tau} \frac{B(y)^p}{V(y)} \right) \right]^{r/p} w(x)\, dx \right)^{1/r},$$

$$F_2 := \left(\int_0^\infty \left(\int_x^\infty \left[\sup_{t \leqslant \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q w(t)\, dt \right)^{r/p} \left[\sup_{0 < \tau \leqslant x} \frac{B^p(\tau)}{V(\tau)} \right]^{r/p} \right.$$
$$\left. \times \left[\sup_{x \leqslant \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q w(x)\, dx \right)^{1/r},$$

$$F_3 := \left(\int_0^\infty W^{r/p}(x) \left(\sup_{x \leqslant \tau < \infty} \left[\sup_{\tau \leqslant y < \infty} \frac{u^p(y)}{V^2(y)} \right] V(\tau) \right)^{r/p} w(x)\, dx \right)^{1/r},$$

$$F_4 := \left(\int_0^\infty \left(\int_x^\infty \left[\sup_{t \leqslant y < \infty} \frac{u^p(y)}{V^2(y)} \right]^{q/p} w(t)\, dt \right)^{r/p} \left[\sup_{x \leqslant y < \infty} \frac{u^p(y)}{V^2(y)} \right]^{q/p} V^{r/p}(x) w(x)\, dx \right)^{1/r},$$

and in this case $C \approx F_1 + F_2 + F_3 + F_4$.

We recall the following results from [27]. Our formulations of these statements are not exactly the same as in the mentioned paper. But by following the proof of these theorems in [27], it is not difficult to see that such formulations are also true.

THEOREM 2.4. *Let $0 < \beta \leqslant \infty$ and $1 \leqslant s < \infty$, and let $T : \mathfrak{M}^+(0,\infty) \to \mathfrak{M}^+(0,\infty)$ satisfying the following conditions:*

(i) *$T(\lambda f) = \lambda T f$ for all $\lambda \geqslant 0$ and $f \in \mathfrak{M}^+(0,\infty)$;*

(ii) *$Tf(x) \leqslant cTg(x)$ for almost all $x \in \mathbb{R}_+$ if $f(x) \leqslant g(x)$ for almost all $x \in \mathbb{R}_+$, with constant $c > 0$ independent of f and g;*

(iii) *$T(f+g) \leqslant c(Tf + Tg)$ for all $f, g \in \mathfrak{M}^+(0,\infty)$, with a constant $c > 0$ independent of f and g.*

Then the inequality

$$\|Tf\|_{\beta,w,(0,\infty)} \leqslant c\|f\|_{s,v,(0,\infty)}, \quad f \in \mathfrak{M}^+((0,\infty);\downarrow) \tag{2.1}$$

holds iff both inequalities

$$\left\| T\left(\int_x^\infty h\right)\right\|_{\beta,w,(0,\infty)} \leqslant c\|h\|_{s,V^s v^{1-s},(0,\infty)}, \quad h \in \mathfrak{M}^+(0,\infty) \tag{2.2}$$

and

$$\|T\mathbf{1}\|_{\beta,w,(0,\infty)} \leqslant c\|\mathbf{1}\|_{s,v,(0,\infty)} \tag{2.3}$$

hold.

THEOREM 2.5. *Let $0 < \beta \leqslant \infty$ and $1 \leqslant s < \infty$, and let $T : \mathfrak{M}^+(0,\infty) \to \mathfrak{M}^+(0,\infty)$ satisfies conditions* (i) - (iii). *Then inequality* (2.1) *holds iff the inequality*

$$\left\| T\left(\frac{1}{V^2(x)}\int_0^x hV\right)\right\|_{\beta,w,(0,\infty)} \leqslant c\|h\|_{s,v^{1-s},(0,\infty)}, \quad h \in \mathfrak{M}^+(0,\infty) \tag{2.4}$$

holds.

Now we give the solution of inequality (1.2), when $p = \infty$ or $q = \infty$.

THEOREM 2.6. *Let $0 < p < \infty$. Assume that $b \in \mathscr{W}(0,\infty)$, $u, w \in \mathscr{W}(0,\infty) \cap C(0,\infty)$ is such that $0 < B(x) < \infty$ and $0 < V(x) < \infty$ for all $x > 0$. Then inequality*

$$\|T_{u,b} f\|_{\infty,w,(0,\infty)} \leqslant C\|f\|_{p,v,(0,\infty)}, \qquad f \in \mathfrak{M}^+((0,\infty);\downarrow) \tag{2.5}$$

is satisfied with the best constant C if and only if the following holds:

(i) *$1 < p$ and $G_1 + G_2 < \infty$, where*

$$G_1 := \sup_{x>0}\left(\sup_{x\leqslant t<\infty}\left[\sup_{0<\tau\leqslant t} w(\tau)\right]\frac{u(t)}{B(t)}\right)\left(\int_0^x \left(\frac{B(y)}{V(y)}\right)^{p'} v(y)\,dy\right)^{1/p'},$$

$$G_2 := \sup_{x>0}\left(\sup_{x\leqslant t<\infty}\left[\sup_{0<\tau\leqslant t} w(\tau)\right]\frac{u(t)}{V^2(t)}\right)\left(\int_0^x V^{p'}(y)v(y)\,dy\right)^{1/p'},$$

and in this case $C \approx G_1 + G_2$;

(ii) $p \leqslant 1$ *and* $H_1 + H_2 < \infty$, *where*

$$H_1 := \sup_{x>0} \left(\sup_{0<y\leqslant x} \left(B(y) \sup_{y\leqslant t<\infty} \left[\sup_{0<\tau\leqslant t} w(\tau) \right] \frac{u(t)}{B(t)} \right) \right) V^{-1/p}(x),$$

$$H_2 := \sup_{x>0} \left(\sup_{x\leqslant t<\infty} \left[\sup_{0<\tau\leqslant t} w(\tau) \right] \frac{u(t)}{B(t)} \right) \frac{B(x)}{V^{1/p}(x)},$$

and in this case $C \approx H_1 + H_2$.

Proof. Whenever F, G are non-negative measurable functions on $(0,\infty)$ and F is non-increasing, then

$$\operatorname*{ess\,sup}_{t\in(0,\infty)} F(t)G(t) = \operatorname*{ess\,sup}_{t\in(0,\infty)} F(t) \operatorname*{ess\,sup}_{\tau\in(0,t)} G(\tau);$$

likewise, when F is non-decreasing, then

$$\operatorname*{ess\,sup}_{t\in(0,\infty)} F(t)G(t) = \operatorname*{ess\,sup}_{t\in(0,\infty)} F(t) \operatorname*{ess\,sup}_{\tau\in(t,\infty)} G(\tau).$$

Hence

$$\begin{aligned}
\|T_{u,b}f\|_{\infty,w,(0,\infty)} &= \sup_{x>0} w(x) \sup_{x\leqslant\tau<\infty} \frac{u(\tau)}{B(\tau)} \int_0^\tau f(y)b(y)\,dy \\
&= \sup_{x>0} \left(\sup_{0<\tau\leqslant x} w(\tau) \right) \sup_{x\leqslant\tau<\infty} \frac{u(\tau)}{B(\tau)} \int_0^\tau f(y)b(y)\,dy \\
&= \sup_{x>0} \left(\sup_{0<\tau\leqslant x} w(\tau) \right) \frac{u(x)}{B(x)} \int_0^x f(y)b(y)\,dy \\
&= \sup_{x>0} \widetilde{w}(x) \int_0^x f(y)b(y)\,dy,
\end{aligned} \tag{2.6}$$

where

$$\widetilde{w}(x) := \left(\sup_{0<\tau\leqslant x} w(\tau) \right) \frac{u(x)}{B(x)} \qquad (x>0),$$

and inequality (2.5) is equivalent to the inequality

$$\sup_{x>0} \widetilde{w}(x) \int_0^x f(y)b(y)\,dy \leqslant C \left(\int_0^\infty f^p(y)v(y)\,dy \right)^{1/p}, \quad f \in \mathfrak{M}^+((0,\infty);\downarrow). \tag{2.7}$$

(i) Let $p>1$. By Theorem 2.4, (2.7) holds iff both

$$\sup_{x>0} \widetilde{w}(x) \int_0^x \left(\int_y^\infty h(\tau)d\tau \right) b(y)dy \leqslant C \left(\int_0^\infty h^p(y)V^p(y)v^{1-p}(y)dy \right)^{1/p}, \; h \in \mathfrak{M}^+(0,\infty), \tag{2.8}$$

and

$$\sup_{x>0} \widetilde{w}(x) B(x) \leqslant C \left(\int_0^\infty v(y)\,dy \right)^{1/p} \tag{2.9}$$

hold.

Evidently, inequality (2.8) is equivalent to the following inequalities:

$$\begin{aligned} &\sup_{x>0} \widetilde{w}(x) \int_0^x h(\tau) B(\tau)\,d\tau \\ &\qquad \leqslant C \left(\int_0^\infty h^p(y) V^p(y) v^{1-p}(y)\,dy \right)^{1/p}, \qquad h \in \mathfrak{M}^+(0,\infty), \end{aligned} \tag{2.10}$$

$$\begin{aligned} &\sup_{x>0} \widetilde{w}(x) B(x) \int_x^\infty h(\tau)\,d\tau \\ &\qquad \leqslant C \left(\int_0^\infty h^p(y) V^p(y) v^{1-p}(y)\,dy \right)^{1/p}, \qquad h \in \mathfrak{M}^+(0,\infty). \end{aligned} \tag{2.11}$$

By Theorem 2.4, inequalities (2.11) and (2.9) hold if and only if the inequality

$$\sup_{x>0} \widetilde{w}(x) B(x) f(x) \leqslant C \left(\int_0^\infty f^p(y) v(y)\,dy \right)^{1/p}, \qquad f \in \mathfrak{M}^+((0,\infty);\downarrow) \tag{2.12}$$

holds.

By Theorem 2.5, inequality (2.12) holds if and only if the inequality

$$\begin{aligned} &\sup_{x>0} \widetilde{w}(x) B(x) \frac{1}{V^2(x)} \int_0^x h(\tau) V(\tau)\,d\tau \\ &\qquad \leqslant C \left(\int_0^\infty h^p(y) v^{1-p}(y)\,dy \right)^{1/p}, \qquad h \in \mathfrak{M}^+(0,\infty) \end{aligned} \tag{2.13}$$

holds.

Consequently, we have shown that (2.5) is equivalent to the following two inequalities:

$$\begin{aligned} &\sup_{x>0} \left[\sup_{0<\tau\leqslant x} w(\tau) \right] \frac{u(x)}{B(x)} \int_0^x h(y)dy \\ &\qquad \leqslant C \left(\int_0^\infty h^p(y) \left(\frac{V(y)}{B(y)} \right)^p v^{1-p}(y)dy \right)^{1/p}, \quad h \in \mathfrak{M}^+(0,\infty), \\ &\sup_{x>0} \left[\sup_{0<\tau\leqslant x} w(\tau) \right] \frac{u(x)}{V^2(x)} \int_0^x h(y)\,dy \\ &\qquad \leqslant C \left(\int_0^\infty \left(\frac{h(y)}{V(y)} \right)^p v^{1-p}(y)dy \right)^{1/p}, \quad h \in \mathfrak{M}^+(0,\infty), \end{aligned}$$

which hold if and only if $G_1 < \infty$ and $G_2 < \infty$, respectively (see, for instance, [41, 33, 34]).

(ii) Let $p \leqslant 1$. It is known that inequality (2.7) holds if and only if

$$\sup_{x>0}\left(\sup_{y>0} B(\min\{x,y\})\left(\sup_{y\leqslant t<\infty}\left[\sup_{0<\tau\leqslant t} w(\tau)\right]\frac{u(t)}{B(t)}\right)\right)V^{-1/p}(x)<\infty$$

(see, for instance, [23, Theorem 5.1, (v)]), which is evidently holds iff $H_1<\infty$ and $H_2<\infty$. □

THEOREM 2.7. *Assume that* $b\in\mathscr{W}(0,\infty)$, $u,w\in\mathscr{W}(0,\infty)\cap C(0,\infty)$ *is such that* $0<B(x)<\infty$ *for all* $x>0$. *Then inequality*

$$\|T_{u,b}f\|_{\infty,w,(0,\infty)}\leqslant C\|f\|_{\infty,v,(0,\infty)},\qquad f\in\mathfrak{M}^{\downarrow}(0,\infty) \tag{2.14}$$

holds if and only if

$$I:=\sup_{x>0}\left(\int_0^x\frac{b(y)\,dy}{\operatorname{ess\,sup}_{\tau\in(0,y)}v(\tau)}\right)\left[\sup_{0<\tau\leqslant x}w(\tau)\right]\frac{u(x)}{B(x)}<\infty.$$

Moreover, the best constant C *in* (2.14) *satisfies* $C\approx I$.

Proof. By (2.6), we know that inequality (2.14) is equivalent to the inequality

$$\begin{aligned}\sup_{x>0}\left(\sup_{x\leqslant t<\infty}\left[\sup_{0<\tau\leqslant t}w(\tau)\right]\frac{u(t)}{B(t)}\right)\int_0^x f(y)b(y)dy&\\ \leqslant C\operatorname*{ess\,sup}_{x>0}f(x)v(x),\quad f\in\mathfrak{M}^+((0,\infty);\downarrow),&\end{aligned} \tag{2.15}$$

which, by [27, Theorem 3.16], holds if and only if

$$\sup_{x>0}\left(\int_0^x\frac{b(y)\,dy}{\operatorname{ess\,sup}_{\tau\in(0,y)}v(\tau)}\right)\left[\sup_{0<\tau\leqslant x}w(\tau)\right]\frac{u(x)}{B(x)}<\infty. \quad \square$$

THEOREM 2.8. *Let* $0<q<\infty$ *and let* $u\in\mathscr{W}(0,\infty)\cap C(0,\infty)$. *Assume that* $b,v,w\in\mathscr{W}(0,\infty)$ *is such that* $0<B(x)<\infty$ *for all* $x>0$. *Then inequality*

$$\|T_{u,b}f\|_{q,w,(0,\infty)}\leqslant c\|f\|_{\infty,v,(0,\infty)},\qquad f\in\mathfrak{M}^+((0,\infty);\downarrow) \tag{2.16}$$

is satisfied with the best constant C *if and only if*

$$J:=\left(\int_0^\infty\left[\sup_{t\leqslant\tau<\infty}\frac{u(\tau)}{B(\tau)}\int_0^\tau\frac{b(y)\,dy}{\operatorname{ess\,sup}_{\tau\in(0,y)}v(\tau)}\right]^q w(x)\,dx\right)^{1/q}<\infty.$$

Moreover, the best constant C *in* (2.14) *satisfies* $C\approx J$.

Proof. The statement follows by [27, Theorem 3.16]. □

3. Main results

In this section we give statements and proofs of our main results.

Let F be any non-negative set function defined on the collection of all sets of positive finite measure. Define its maximal function by

$$MF(x) := \sup_{Q\ni x} F(Q),$$

where the supremum is taken over all cubes containing x.

DEFINITION 3.1. [36, Definition 1] We say that a set function F is pseudo-increasing if there is a positive constant $C > 0$ such that for any finite collection of pairwise disjoint cubes $\{Q_j\}$, we have

$$\min_i F(Q_i) \leqslant CF\left(\bigcup_i Q_i\right). \tag{3.1}$$

THEOREM 3.2. [36, Theorem 1] *Let F be a pseudo-increasing set function. Then, for any $t > 0$,*

$$(MF)^*(t) \leqslant C \sup_{|E|>t/3^n} F(E), \tag{3.2}$$

where C is the constant appearing in (3.1), and the supremum is taken over all sets E of finite measure $|E| > t/3^n$.

LEMMA 3.3. *Let $0 < r < \infty$. Assume that $\phi \in Q_r$. Suppose that X is a quasi-Banach function space on a measure space $(\mathbb{R}^n, dx)$. Moreover, assume that X satisfy a lower r-estimate. Then there exists $C > 0$ such that for any function f from X and any finite pairwise disjoint collection cubes $\{Q_j\}$ on $\mathbb{R}^n$*

$$\min_i \frac{\|f\chi_{Q_i}\|_X}{\phi(|Q_i|)} \leqslant C \frac{\|f\chi_{\cup_i Q_i}\|_X}{\phi(|\cup_i Q_i|)} \tag{3.3}$$

holds true.

Proof. Denote by

$$A := \min_i \frac{\|f\chi_{Q_i}\|_X}{\phi(|Q_i|)}.$$

Since $\phi \in Q_r$, we have that

$$A\phi(|\cup_i Q_i|) = A\phi\left(\sum_i |Q_i|\right) \lesssim A\left(\sum_i \phi(|Q_i|)^r\right)^{1/r} \leqslant \left(\sum_i \|f\chi_{Q_i}\|_X^r\right)^{1/r}.$$

On using the r-lower estimate property of X, we get that

$$A\phi(|\cup_i Q_i|) \lesssim \left\|\sum_{i=1}^n f\chi_{Q_i}\right\|_X = \|f\chi_{\cup_i Q_i}\|_X. \quad \square$$

LEMMA 3.4. *Let $0 < r < \infty$. Assume that $\phi \in Q_r$. Suppose that X is a quasi-Banach function space satisfying a lower r-estimate. Then, for any $t > 0$,*

$$(M_{\phi,X} f)^*(t) \leqslant C \sup_{|E|>t/3^n} \frac{\|f\chi_E\|_X}{\phi(|E|)} \tag{3.4}$$

holds, where $C > 0$ is the constant appearing in (3.3).

Proof. The statement follows by Theorem 3.2 and Lemma 3.3. □

LEMMA 3.5. *Let $0 < r < \infty$. Assume that $\phi \in Q_r$. Suppose that X is a r.i. quasi-Banach function space satisfying a lower r-estimate. Then, for any $t > 0$,*

$$(M_{\phi,X} f)^*(t) \leqslant C \sup_{\tau>t} \frac{\|f^*\chi_{[0,\tau)}\|_{\bar{X}}}{\phi(\tau)} \tag{3.5}$$

holds, where $C > 0$ is constant independent of f and t.

Proof. By Lemma 3.4, we have that

$$\begin{aligned}
(M_{\phi,X} f)^*(t) &\leqslant C \sup_{|E|>t/3^n} \frac{\|f\chi_E\|_X}{\phi(|E|)} \\
&= C \sup_{|E|>t/3^n} \frac{\|(f\chi_E)^*\|_{\bar{X}}}{\phi(|E|)} \\
&\leqslant C \sup_{|E|>t/3^n} \frac{\|f^*\chi_{[0,|E|)}\|_{\bar{X}}}{\phi(|E|)} \\
&\leqslant C \sup_{\tau>t/3^n} \frac{\|f^*\chi_{[0,\tau)}\|_{\bar{X}}}{\phi(\tau)}.
\end{aligned}$$

Since $\phi \in Q_r$ implies $\phi \in \Delta_2$, we obtain that

$$\sup_{\tau>t/3^n} \frac{\|f^*\chi_{[0,\tau)}\|_{\bar{X}}}{\phi(\tau)} \leqslant C \sup_{3^n\tau>t} \frac{\|f^*\chi_{[0,3^n\tau)}\|_{\bar{X}}}{\phi(3^n\tau)} = C \sup_{\tau>t} \frac{\|f^*\chi_{[0,\tau)}\|_{\bar{X}}}{\phi(\tau)}.$$

Combining, we arrive at (3.5). □

COROLLARY 3.6. *Let $0 < \alpha \leqslant r < \infty$, $\phi \in Q_r$ and $b \in \mathscr{W}(0,\infty)$ be such that $B(\infty) = \infty$, $B \in \Delta_2$ and $B(t)/t^{\alpha/r}$ is quasi-increasing. Then there exists a constant $C > 0$ such that for any measurable function f on $\mathbb{R}^n$ the inequality*

$$(M_{\phi,\Lambda^\alpha(b)} f)^*(t) \leqslant C \sup_{\tau>t} \frac{\left(\int_0^\tau (f^*)^\alpha(y) b(y)\,dy\right)^{1/\alpha}}{\phi(\tau)}$$

holds.

Proof. In view of Theorem 1.1, $\Lambda^\alpha(b)$ satisfies a lower r-estimate. Then the statement follows from Lemma 3.5, when $X=\Lambda^\alpha(b)$. $\square$

COROLLARY 3.7. *Let $0<q\leqslant p<\infty$. Then there exists a constant $C>0$ such that for any measurable function f on $\mathbb{R}^n$ the inequality*

$$(M_{p,q}f)^*(t)\leqslant \frac{C}{t^{1/p}}\left(\int_0^t (f^*)^q(y)y^{q/p-1}\,dy\right)^{1/q} \tag{3.6}$$

holds.

Proof. Let $\alpha=q$, $b(t)=t^{q/p-1}$ and $\phi(t)=t^{1/p}$ $(t>0)$. Then $M_{p,q}=M_{\phi,\Lambda^\alpha(b)}$. It is clear that $B(t)\approx t^{q/p}$ $(t>0)$. Since $\phi\in Q_r$, $B\in\Delta_2$ and $B(t)/t^{q/r}$ is quasi-increasing when $r=p\geqslant q$, by Corollary 3.6, we get that

$$(M_{p,q}f)^*(t)\leqslant C\sup_{\tau>t}\frac{1}{\tau^{1/p}}\left(\int_0^\tau (f^*)^q(y)y^{q/p-1}\,dy\right)^{1/q}.$$

It is easy to see that function $G(\tau)=\frac{1}{\tau^{q/p}}\int_0^\tau g(y)dy^{q/p}$ is non-increasing on $(0,\infty)$ when g is non-increasing. Consequently,

$$\sup_{\tau>t}\frac{1}{\tau^{1/p}}\left(\int_0^\tau (f^*)^q(y)y^{q/p-1}\,dy\right)^{1/q}=\frac{1}{t^{1/p}}\left(\int_0^t (f^*)^q(y)y^{q/p-1}\,dy\right)^{1/q}.$$

Thus

$$(M_{p,q}f)^*(t)\leqslant \frac{C}{t^{1/p}}\left(\int_0^t (f^*)^q(y)y^{q/p-1}\,dy\right)^{1/q}. \quad \square$$

REMARK 3.8. Note that inequality (3.6) was proved in [2] with the help of interpolation. This result was extended to more general setting of maximal operators in [38].

REMARK 3.9. It is clear that if $\omega\in Q_r$, $0<r<\infty$ and $g:(0,\infty)\to(0,\infty)$ is a quasi-decreasing function, then $\omega\cdot g\in Q_r$. Indeed: Since $g\big(\sum_{i=1}^n t_i\big)\leqslant C\min_i g(t_i)$, we get that

$$\begin{aligned}
(\omega\cdot g)\left(\sum_{i=1}^n t_i\right)&=\omega\left(\sum_{i=1}^n t_i\right)\cdot g\left(\sum_{i=1}^n t_i\right)\\
&\leqslant C\left(\sum_{i=1}^n \omega(t_i)^r\right)^{1/r}\cdot\min_i g(t_i)\\
&=C\left(\sum_{i=1}^n \big(\omega(t_i)\cdot\min_i g(t_i)\big)^r\right)^{1/r}\\
&\leqslant C\left(\sum_{i=1}^n (\omega\cdot g)(t_i)^r\right)^{1/r}.
\end{aligned}$$

COROLLARY 3.10. *Let $s \in (0,\infty)$, $\gamma \in (0,n)$ and $\mathbb{A} = (A_0, A_\infty) \in \mathbb{R}^2$. Then there exists a constant $C > 0$ depending only in n, s, γ and $\mathbb{A}$ such that for all $f \in \mathfrak{M}(\mathbb{R}^n)$ and every $t \in (0,\infty)$*

$$(M_{s,\gamma,\mathbb{A}} f)^*(t) \leqslant C \left[\sup_{\tau > t} \tau^{\gamma/n-1} \ell^{-s\mathbb{A}}(\tau) \int_0^\tau (f^*)^s(y)\,dy \right]^{1/s}. \tag{3.7}$$

Proof. It is mentioned in the introduction that $M_{\phi,\Lambda^\alpha(b)} \approx M_{s,\gamma,\mathbb{A}}$, when $\alpha = s$, $b \equiv 1$ and $\phi(t) = t^{(n-\gamma)/(sn)} \ell^{\mathbb{A}}(t)$, $(t > 0)$. Let $r = s$. Writing $\phi = \omega \cdot g$, where $\omega(t) = t^{1/s}$ and $g(t) = t^{-\gamma/(sn)} \ell^{\mathbb{A}}(t)$, $(t > 0)$, observing that $\omega \in Q_s$ and g is quasi-decreasing, in view of remark 3.9, we claim that $\phi \in Q_r$. On the other side, since $B(t) = t$, $t > 0$, we get that $B \in \Delta_2$ and $B(t)/t^{\alpha/r} \equiv 1$ is quasi-increasing. Hence, by Corollary 3.6, inequality (3.7) holds. $\square$

REMARK 3.11. Note that inequality (3.7) was proved in [18, Theorem 3.1].

LEMMA 3.12. *Let $0 < r < \infty$. Assume that $\phi \in \Delta_2$ is a quasi-increasing function on $(0,\infty)$. Suppose that X is a r.i. quasi-Banach function space. Then, for any $t > 0$,*

$$(M_{\phi,X} f)^*(t) \geqslant c \sup_{\tau > t} \frac{\|f^* \chi_{[0,\tau)}\|_{\bar{X}}}{\phi(\tau)}, \qquad f \in \mathfrak{M}^{\mathrm{rad},\downarrow}(\mathbb{R}^n) \tag{3.8}$$

holds, where $c > 0$ is constant independent of f and t.

Proof. Let f be any function from $\mathfrak{M}^{\mathrm{rad},\downarrow}$. For every $x, y \in \mathbb{R}^n$ such that $|y| > |x|$, we have that

$$(M_{\phi,X} f)(x) \gtrsim \frac{\|f \chi_{B(0,|y|)}\|_X}{\phi(|B(0,|y|)|)}.$$

Since $(f \chi_{B(0,|y|)})^*(t) = f^*(t) \chi_{[0,|B(0,|y|)|)}(t)$, $t > 0$, we get that

$$(M_{\phi,X} f)(x) \gtrsim \frac{\|f^* \chi_{[0,|B(0,|y|)|)}\|_{\bar{X}}}{\phi(|B(0,|y|)|)}.$$

Hence

$$(M_{\phi,X} f)(x) \gtrsim \sup_{|y|>|x|} \frac{\|f^* \chi_{[0,|B(0,|y|)|)}\|_{\bar{X}}}{\phi(|B(0,|y|)|)} = \sup_{|y|>|x|} \frac{\|f^* \chi_{[0,\omega_n |y|^n)}\|_{\bar{X}}}{\phi(\omega_n |y|^n)} = \sup_{\tau > \omega_n |x|^n} \frac{\|f^* \chi_{[0,\tau)}\|_{\bar{X}}}{\phi(\tau)}$$

holds, where ω_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

Recall that

$$f^*(t) = \sup_{|E|=t} \operatorname*{ess\,inf}_{x \in E} |f(x)|, \quad t \in (0,\infty),$$

(see, for instance, [14, p. 33]).

On taking rearrangements, we obtain that

$$\begin{aligned}
(M_{\phi,X}f)^*(t) &= \sup_{|E|=t} \operatorname*{ess\,inf}_{x\in E} (M_{\phi,X}f)(x) \\
&\geqslant \operatorname*{ess\,inf}_{x\in B(0,(t/\omega_n)^{1/n})} (M_{\phi,X}f)(x) \\
&\gtrsim \operatorname*{ess\,inf}_{x\in B(0,(t/\omega_n)^{1/n})} \sup_{\tau>\omega_n|x|^n} \frac{\|f^*\chi_{[0,\tau)}\|_{\bar{X}}}{\phi(\tau)} \\
&= \operatorname*{ess\,inf}_{0\leqslant s<t} \sup_{\tau>s} \frac{\|f^*\chi_{[0,\tau)}\|_{\bar{X}}}{\phi(\tau)} = \sup_{\tau>t} \frac{\|f^*\chi_{[0,\tau)}\|_{\bar{X}}}{\phi(\tau)}. \qquad \square
\end{aligned}$$

We are now in a position to prove our main result.

Proof of Theorem 1.2. The statement follows by Lemmas 3.5 and 3.12. $\square$

3.1. Boundedness of $M_{\phi,\Lambda^\alpha(b)} : \Lambda^p(v) \to \Lambda^q(w)$, $0<p,q<\infty$

THEOREM 3.13. *Let $0<p,q<\infty$, $0<\alpha\leqslant r<\infty$ and $v,w\in\mathscr{W}(0,\infty)$. Assume that $\phi\in Q_r$ is a quasi-increasing function. Moreover, assume that $b\in\mathscr{W}(0,\infty)$ is such that $0<B(t)<\infty$ for all $t>0$, $B(\infty)=\infty$, $B\in\Delta_2$ and $B(t)/t^{\alpha/r}$ is quasi-increasing. Then $M_{\phi,\Lambda^\alpha(b)}$ is bounded from $\Lambda^p(v)$ to $\Lambda^q(w)$, that is, the inequality*

$$\|M_{\phi,\Lambda^\alpha(b)}\|_{\Lambda^q(w)} \leqslant C\|f\|_{\Lambda^p(v)}$$

holds for all $f\in\mathfrak{M}(\mathbb{R}^n)$ if and only if the inequality

$$\|T_{B/\phi^\alpha,b}\psi\|_{q/\alpha,w,(0,\infty)} \leqslant C^\alpha \|\psi\|_{p/\alpha,v,(0,\infty)} \tag{3.9}$$

holds for all $\psi\in\mathfrak{M}^+((0,\infty);\downarrow)$.

Proof. The statement follows from Theorem 1.2, (a), when $X=\Lambda^\alpha(b)$. $\square$

THEOREM 3.14. *Let $0<p,q<\infty$, $0<\alpha\leqslant r<\infty$ and $v,w\in\mathscr{W}(0,\infty)$. Assume that $\phi\in Q_r$ is a quasi-increasing function. Moreover, assume that $b\in\mathscr{W}(0,\infty)$ is such that $0<B(t)<\infty$ for all $t>0$, $B(\infty)=\infty$, $B\in\Delta_2$ and $B(t)/t^{\alpha/r}$ is quasi-increasing. Then $M_{\phi,\Lambda^\alpha(b)}$ is bounded from $\Lambda^p(v)$ to $\Lambda^q(w)$ if and only if the following holds:*

(i) *$\alpha<p\leqslant q$ and $\mathscr{A}_1+\mathscr{A}_2<\infty$, where*

$$\mathscr{A}_1 := \sup_{x>0}\left(\phi^{-q}(x)W(x)+\int_x^\infty \phi^{-q}(t)w(t)\,dt\right)^{\frac{1}{q}}\left(\int_0^x \left(\frac{B(y)}{V(y)}\right)^{\frac{p}{p-\alpha}} v(y)\,dy\right)^{\frac{p-\alpha}{p\alpha}},$$

$$\begin{aligned}
\mathscr{A}_2 := \sup_{x>0}&\left(\left[\sup_{x\leqslant\tau<\infty}\frac{B(\tau)}{\phi^\alpha(\tau)V^2(\tau)}\right]^{\frac{q}{\alpha}} W(x)+\int_x^\infty\left[\sup_{t\leqslant\tau<\infty}\frac{B(\tau)}{\phi^\alpha(\tau)V^2(\tau)}\right]^{\frac{q}{\alpha}} w(t)\,dt\right)^{\frac{1}{q}} \\
&\times\left(\int_0^x V^{\frac{p}{p-\alpha}} v\right)^{\frac{p-\alpha}{p\alpha}},
\end{aligned}$$

and in this case $\|M_{\phi,\Lambda^\alpha(b)}\|_{\Lambda^p(v)\to\Lambda^q(w)} \approx \mathscr{A}_1+\mathscr{A}_2$;

(ii) $\alpha = p \leqslant q$ *and* $\mathscr{B}_1+\mathscr{B}_2<\infty$, *where*

$$\mathscr{B}_1 := \sup_{x>0}\left(\phi^{-q}(x)W(x)+\int_x^\infty \phi^{-q}(t)w(t)\,dt\right)^{\frac{1}{q}}\left(\sup_{0<y\leqslant x}\frac{B(y)}{V(y)}\right)^{\frac{1}{\alpha}},$$

$$\mathscr{B}_2 := \sup_{x>0}\left(\left[\sup_{x\leqslant\tau<\infty}\frac{B(\tau)}{\phi^\alpha(\tau)V^2(\tau)}\right]^{\frac{q}{\alpha}}W(x)+\int_x^\infty\left[\sup_{t\leqslant\tau<\infty}\frac{B(\tau)}{\phi^\alpha(\tau)V^2(\tau)}\right]^{\frac{q}{\alpha}}w(t)dt\right)^{\frac{1}{q}}V^{\frac{1}{\alpha}}(x),$$

and in this case $\|M_{\phi,\Lambda^\alpha(b)}\|_{\Lambda^p(v)\to\Lambda^q(w)} \approx \mathscr{B}_1+\mathscr{B}_2$;

(iii) $\max\{\alpha,q\}<p$ *and* $\mathscr{C}_1+\mathscr{C}_2+\mathscr{C}_3+\mathscr{C}_4<\infty$, *where*

$$\mathscr{C}_1 := \left(\int_0^\infty\left(\int_x^\infty \phi^{-q}(t)w(t)\,dt\right)^{\frac{q}{p-q}}\phi^{-q}(x)\right.$$
$$\left.\times\left(\int_0^x\left(\frac{B(y)}{V(y)}\right)^{\frac{p}{p-\alpha}}v(y)\,dy\right)^{\frac{q(p-\alpha)}{\alpha(p-q)}}w(x)\,dx\right)^{\frac{p-q}{pq}},$$

$$\mathscr{C}_2 := \left(\int_0^\infty W^{\frac{q}{p-q}}(x)\left[\sup_{x\leqslant\tau<\infty}\phi^{-\alpha}(\tau)\left(\int_0^\tau\left(\frac{B(y)}{V(y)}\right)^{\frac{p}{p-\alpha}}v(y)\,dy\right)^{\frac{p-\alpha}{p}}\right]^{\frac{pq}{\alpha(p-q)}}w(x)\,dx\right)^{\frac{p-q}{pq}},$$

$$\mathscr{C}_3 := \left(\int_0^\infty\left(\int_x^\infty\left[\sup_{t\leqslant\tau<\infty}\frac{B(\tau)}{\phi^\alpha(\tau)V^2(\tau)}\right]^{\frac{q}{\alpha}}w(t)\,dt\right)^{\frac{q}{p-q}}\left[\sup_{x\leqslant\tau<\infty}\frac{B(\tau)}{\phi^\alpha(\tau)V^2(\tau)}\right]^{\frac{q}{\alpha}}\right.$$
$$\left.\times\left(\int_0^x V^{\frac{p}{p-\alpha}}v\right)^{\frac{q(p-\alpha)}{\alpha(p-q)}}w(x)\,dx\right)^{\frac{p-q}{pq}},$$

$$\mathscr{C}_4 := \left(\int_0^\infty W^{\frac{q}{p-q}}(x)\left[\sup_{x\leqslant\tau<\infty}\left[\sup_{\tau\leqslant y<\infty}\frac{B(y)}{\phi^\alpha(y)V^2(y)}\right]\left(\int_0^\tau V^{\frac{p}{p-\alpha}}v\right)^{\frac{p-\alpha}{p}}\right]^{\frac{pq}{\alpha(p-q)}}w(x)\,dx\right)^{\frac{p-q}{pq}},$$

and in this case $\|M_{\phi,\Lambda^\alpha(b)}\|_{\Lambda^p(v)\to\Lambda^q(w)} \approx \mathscr{C}_1+\mathscr{C}_2+\mathscr{C}_3+\mathscr{C}_4$;

(iv) $q<\alpha=p$ *and* $\mathscr{D}_1+\mathscr{D}_2+\mathscr{D}_3+\mathscr{D}_4<\infty$, *where*

$$\mathscr{D}_1 := \left(\int_0^\infty\left(\int_x^\infty \phi^{-q}(t)w(t)\,dt\right)^{\frac{q}{p-q}}\phi^{-q}(x)\left(\sup_{0<y\leqslant x}\frac{B(y)}{V(y)}\right)^{\frac{pq}{\alpha(p-q)}}w(x)\,dx\right)^{\frac{p-q}{pq}},$$

$$\mathscr{D}_2 := \left(\int_0^\infty W^{\frac{q}{p-q}}(x)\left[\sup_{x\leqslant\tau<\infty}\phi^{-\alpha}(\tau)\left(\sup_{0<y\leqslant\tau}\frac{B(y)}{V(y)}\right)\right]^{\frac{pq}{\alpha(p-q)}}w(x)\,dx\right)^{\frac{p-q}{pq}},$$

$$\mathscr{D}_3 := \left(\int_0^\infty\left(\int_x^\infty\left[\sup_{t\leqslant\tau<\infty}\frac{B(\tau)}{\phi^\alpha(\tau)V^2(\tau)}\right]^{\frac{q}{\alpha}}w(t)\,dt\right)^{\frac{q}{p-q}}\right.$$
$$\left.\times\left[\sup_{x\leqslant\tau<\infty}\frac{B(\tau)}{\phi^\alpha(\tau)V^2(\tau)}\right]^{\frac{q}{\alpha}}V^{\frac{pq}{\alpha(p-q)}}(x)w(x)\,dx\right)^{\frac{p-q}{pq}},$$

$$\mathscr{D}_4 := \left(\int_0^\infty W^{\frac{q}{p-q}}(x)\left[\sup_{x\leqslant\tau<\infty}\left[\sup_{\tau\leqslant y<\infty}\frac{B(y)}{\phi^\alpha(y)V^2(y)}\right]V(\tau)\right]^{\frac{pq}{\alpha(p-q)}}w(x)\,dx\right)^{\frac{p-q}{pq}},$$

and in this case $\|M_{\phi,\Lambda^\alpha(b)}\|_{\Lambda^p(v)\to\Lambda^q(w)} \approx \mathscr{D}_1+\mathscr{D}_2+\mathscr{D}_3+\mathscr{D}_4$;

(v) $p \leqslant \min\{\alpha,q\}$ *and* $\mathscr{E}_1+\mathscr{E}_2<\infty$, *where*

$$\mathscr{E}_1 := \sup_{x>0}\left(\phi^{-q}(x)W(x)+\int_x^\infty \phi^{-q}(t)w(t)\,dt\right)^{\frac{1}{q}} \sup_{0<y\leqslant x}\frac{B^{\frac{1}{\alpha}}(y)}{V^{\frac{1}{p}}(y)},$$

$$\mathscr{E}_2 := \sup_{x>0}\left(\left[\sup_{x\leqslant y<\infty}\frac{B^{\frac{1}{\alpha}}(y)}{\phi(y)V^{\frac{2}{p}}(y)}\right]^q W(x)+\int_x^\infty\left[\sup_{t\leqslant y<\infty}\frac{B^{\frac{1}{\alpha}}(y)}{\phi(y)V^{\frac{2}{p}}(y)}\right]^q w(t)\,dt\right)^{\frac{1}{q}} V^{\frac{1}{p}}(x),$$

and in this case $\|M_{\phi,\Lambda^\alpha(b)}\|_{\Lambda^p(v)\to\Lambda^q(w)} \approx \mathscr{E}_1+\mathscr{E}_2$;

(vi) $q<p\leqslant\alpha$ *and* $\mathscr{F}_1+\mathscr{F}_2+\mathscr{F}_3+\mathscr{F}_4<\infty$, *where*

$$\mathscr{F}_1 := \left(\int_0^\infty W^{\frac{q}{p-q}}(x)\left[\sup_{x\leqslant\tau<\infty}\phi^{-q}(\tau)\left(\sup_{0<y\leqslant\tau}\frac{B(y)}{V^{\frac{\alpha}{p}}(y)}\right)\right]^{\frac{pq}{\alpha(p-q)}} w(x)\,dx\right)^{\frac{p-q}{pq}},$$

$$\mathscr{F}_2 := \left(\int_0^\infty\left(\int_x^\infty \phi^{-q}(t)w(t)\,dt\right)^{\frac{q}{p-q}}\left[\sup_{0<\tau\leqslant x}\frac{B(\tau)}{V^{\frac{\alpha}{p}}(\tau)}\right]^{\frac{pq}{\alpha(p-q)}}\phi^{-q}(x)w(x)\,dx\right)^{\frac{p-q}{pq}},$$

$$\mathscr{F}_3 := \left(\int_0^\infty W^{\frac{q}{p-q}}(x)\left(\sup_{x\leqslant\tau<\infty}\left[\sup_{\tau\leqslant y<\infty}\frac{B^{\frac{1}{\alpha}}(y)}{\phi(y)V^{\frac{2}{p}}(y)}\right]V^{\frac{1}{p}}(\tau)\right)^{\frac{pq}{p-q}} w(x)\,dx\right)^{\frac{p-q}{pq}},$$

$$\mathscr{F}_4 := \left(\int_0^\infty\left(\int_x^\infty\left[\sup_{t\leqslant y<\infty}\frac{B^{\frac{1}{\alpha}}(y)}{\phi(y)V^{\frac{2}{p}}(y)}\right]^q w(t)\,dt\right)^{\frac{q}{p-q}}\left[\sup_{x\leqslant y<\infty}\frac{B^{\frac{1}{\alpha}}(y)}{\phi(y)V^{\frac{2}{p}}(y)}\right]^q \times V^{\frac{q}{p-q}}(x)w(x)\,dx\right)^{\frac{p-q}{pq}},$$

and in this case $\|M_{\phi,\Lambda^\alpha(b)}\|_{\Lambda^p(v)\to\Lambda^q(w)} \approx \mathscr{F}_1+\mathscr{F}_2+\mathscr{F}_3+\mathscr{F}_4$.

Proof. The statement follows from Theorems 3.13 and 2.3. $\square$

3.2. Boundedness of $M_{\phi,\Lambda^\alpha(b)} : \Lambda^p(v)\to\Lambda^{q,\infty}(w)$, $0<p,q<\infty$

THEOREM 3.15. *Let* $0<p,q<\infty$, $0<\alpha\leqslant r<\infty$ *and* $v,w\in\mathscr{W}(0,\infty)$. *Assume that* $\phi\in Q_r$ *is a quasi-increasing function. Moreover, assume that* $b\in\mathscr{W}(0,\infty)$ *is such that* $0<B(t)<\infty$ *for all* $t>0$, $B(\infty)=\infty$, $B\in\Delta_2$ *and* $B(t)/t^{\alpha/r}$ *is quasi-increasing. Then* $M_{\phi,\Lambda^\alpha(b)}$ *is bounded from* $\Lambda^p(v)$ *to* $\Lambda^{q,\infty}(w)$, *that is, the inequality*

$$\|M_{\phi,\Lambda^\alpha(b)}\|_{\Lambda^{q,\infty}(w)} \leqslant C\|f\|_{\Lambda^p(v)}$$

holds for all $f\in\mathfrak{M}(\mathbb{R}^n)$ *if and only if the inequality*

$$\|T_{B/\phi^\alpha,b}\psi\|_{\infty,W^{\alpha/q},(0,\infty)} \leqslant C^\alpha\|\psi\|_{p/\alpha,v,(0,\infty)} \tag{3.10}$$

holds for all $\psi\in\mathfrak{M}^+((0,\infty);\downarrow)$.

Proof. The statement follows from Theorem 1.2, (b), when $X = \Lambda^{\alpha}(b)$. $\square$

THEOREM 3.16. *Let $0 < p,q < \infty$, $0 < \alpha \leqslant r < \infty$ and $v,w \in \mathscr{W}(0,\infty)$. Assume that $\phi \in Q_r$ is a quasi-increasing function. Moreover, assume that $b \in \mathscr{W}(0,\infty)$ is such that $0 < B(t) < \infty$ for all $t > 0$, $B(\infty) = \infty$, $B \in \Delta_2$ and $B(t)/t^{\alpha/r}$ is quasi-increasing. Then $M_{\phi,\Lambda^{\alpha}(b)}$ is bounded from $\Lambda^p(v)$ to $\Lambda^{q,\infty}(w)$ if and only if the following holds:*

(i) *$\alpha < p$ and $\mathscr{G}_1 + \mathscr{G}_2 < \infty$, where*

$$\mathscr{G}_1 := \sup_{x>0}\left[\sup_{x\leqslant t<\infty}\frac{W^{\frac{1}{q}}(t)}{\phi(t)}\right]\left(\int_0^x\left(\frac{B(y)}{V(y)}\right)^{\frac{p}{p-\alpha}}v(y)\,dy\right)^{\frac{p-\alpha}{p\alpha}},$$

$$\mathscr{G}_2 := \sup_{x>0}\left[\sup_{x\leqslant t<\infty}\frac{W^{\frac{1}{q}}(t)B^{\frac{1}{\alpha}}(t)}{\phi(t)V^{\frac{2}{\alpha}}(t)}\right]\left(\int_0^x V^{\frac{p}{p-\alpha}}v\right)^{\frac{p-\alpha}{p\alpha}},$$

and in this case $\|M_{\phi,\Lambda^{\alpha}(b)}\|_{\Lambda^p(v)\to\Lambda^{q,\infty}(w)} \approx \mathscr{G}_1 + \mathscr{G}_2$;

(ii) *$p \leqslant \alpha$ and $\mathscr{H}_1 + \mathscr{H}_2 < \infty$, where*

$$\mathscr{H}_1 := \sup_{x>0}\left(\sup_{0<y\leqslant x} B^{\frac{1}{\alpha}}(y)\left[\sup_{y\leqslant t<\infty}\frac{W^{\frac{1}{q}}(t)}{\phi(t)}\right]\right)V^{-\frac{1}{p}}(x),$$

$$\mathscr{H}_2 := \sup_{x>0}\left[\sup_{x\leqslant t<\infty}\frac{W^{\frac{1}{q}}(t)}{\phi(t)}\right]\frac{B^{\frac{1}{\alpha}}(x)}{V^{\frac{1}{p}}(x)},$$

and in this case $\|M_{\phi,\Lambda^{\alpha}(b)}\|_{\Lambda^p(v)\to\Lambda^{q,\infty}(w)} \approx \mathscr{H}_1 + \mathscr{H}_2$.

Proof. The statement follows by Theorems 3.15 and 2.6. $\square$

3.3. Boundedness of $M_{\phi,\Lambda^{\alpha}(b)} : \Lambda^{p,\infty}(v) \to \Lambda^{q,\infty}(w)$, $0 < p, q < \infty$

THEOREM 3.17. *Let $0 < p,q < \infty$, $0 < \alpha \leqslant r < \infty$ and $v,w \in \mathscr{W}(0,\infty)$. Assume that $\phi \in Q_r$ is a quasi-increasing function. Moreover, assume that $b \in \mathscr{W}(0,\infty)$ is such that $0 < B(t) < \infty$ for all $t > 0$, $B(\infty) = \infty$, $B \in \Delta_2$ and $B(t)/t^{\alpha/r}$ is quasi-increasing. Then $M_{\phi,\Lambda^{\alpha}(b)}$ is bounded from $\Lambda^{p,\infty}(v)$ to $\Lambda^{q,\infty}(w)$, that is, the inequality*

$$\|M_{\phi,\Lambda^{\alpha}(b)}\|_{\Lambda^{q,\infty}(w)} \leqslant C\|f\|_{\Lambda^{p,\infty}(v)}$$

holds for all $f \in \mathfrak{M}(\mathbb{R}^n)$ if and only if the inequality

$$\|T_{B/\phi^{\alpha},b}\psi\|_{\infty,W^{\alpha/q},(0,\infty)} \leqslant C^{\alpha}\|\psi\|_{\infty,V^{\alpha/p},(0,\infty)}$$

holds for all $\psi \in \mathfrak{M}^+((0,\infty);\downarrow)$.

Proof. The statement follows from Theorem 1.2, (c), when $X = \Lambda^{\alpha}(b)$. $\square$

THEOREM 3.18. *Let $0 < p,q < \infty$, $0 < \alpha \leqslant r < \infty$ and $v, w \in \mathscr{W}(0,\infty)$. Assume that $\phi \in Q_r$ is a quasi-increasing function. Moreover, assume that $b \in \mathscr{W}(0,\infty)$ is such that $0 < B(t) < \infty$ for all $t > 0$, $B(\infty) = \infty$, $B \in \Delta_2$ and $B(t)/t^{\alpha/r}$ is quasi-increasing. Then $M_{\phi,\Lambda^\alpha(b)}$ is bounded from $\Lambda^{p,\infty}(v)$ to $\Lambda^{q,\infty}(w)$ if and only if*

$$\mathscr{I} := \sup_{x>0} \left(\int_0^x \frac{b(y)}{V^{\frac{\alpha}{p}}(y)}\, dy \right)^{\frac{1}{\alpha}} \frac{W^{\frac{1}{q}}(x)}{\phi(x)} < \infty.$$

Moreover, $\|M_{\phi,\Lambda^\alpha(b)}\|_{\Lambda^{p,\infty}(v)\to\Lambda^{q,\infty}(w)} \approx \mathscr{I}$.

Proof. The statement follows by Theorems 3.17 and 2.7. $\square$

3.4. Boundedness of $M_{\phi,\Lambda^\alpha(b)} : \Lambda^{p,\infty}(v) \to \Lambda^q(w)$, $0 < p, q < \infty$

THEOREM 3.19. *Let $0 < p,q < \infty$, $0 < \alpha \leqslant r < \infty$ and $v, w \in \mathscr{W}(0,\infty)$. Assume that $\phi \in Q_r$ is a quasi-increasing function. Moreover, assume that $b \in \mathscr{W}(0,\infty)$ is such that $0 < B(t) < \infty$ for all $t > 0$, $B(\infty) = \infty$, $B \in \Delta_2$ and $B(t)/t^{\alpha/r}$ is quasi-increasing. Then $M_{\phi,\Lambda^\alpha(b)}$ is bounded from $\Lambda^{p,\infty}(v)$ to $\Lambda^q(w)$, that is, the inequality*

$$\|M_{\phi,\Lambda^\alpha(b)}\|_{\Lambda^q(w)} \leqslant C\|f\|_{\Lambda^{p,\infty}(v)}$$

holds for all $f \in \mathfrak{M}(\mathbb{R}^n)$ if and only if the inequality

$$\|T_{B/\phi^\alpha,b}\psi\|_{q/\alpha,w,(0,\infty)} \leqslant C^\alpha \|\psi\|_{\infty,V^{\alpha/p},(0,\infty)}$$

holds for all $\psi \in \mathfrak{M}^+((0,\infty);\downarrow)$.

Proof. The statement follows from Theorem 1.2, (d), when $X = \Lambda^\alpha(b)$. $\square$

THEOREM 3.20. *Let $0 < p,q < \infty$, $0 < \alpha \leqslant r < \infty$ and $v, w \in \mathscr{W}(0,\infty)$. Assume that $\phi \in Q_r$ is a quasi-increasing function. Moreover, assume that $b \in \mathscr{W}(0,\infty)$ is such that $0 < B(t) < \infty$ for all $t > 0$, $B(\infty) = \infty$, $B \in \Delta_2$ and $B(t)/t^{\alpha/r}$ is quasi-increasing. Then $M_{\phi,\Lambda^\alpha(b)}$ is bounded from $\Lambda^{p,\infty}(v)$ to $\Lambda^q(w)$ if and only if*

$$\mathscr{J} := \left(\int_0^\infty \left(\sup_{x\leqslant\tau<\infty} \frac{1}{\phi^\alpha(\tau)} \int_0^\tau \frac{b(y)}{V^{\frac{\alpha}{p}}(y)}\, dy \right)^{\frac{q}{\alpha}} w(x)\, dx \right)^{\frac{1}{q}} < \infty.$$

Moreover, $\|M_{\phi,\Lambda^\alpha(b)}\|_{\Lambda^{p,\infty}(v)\to\Lambda^q(w)} \approx \mathscr{J}$.

Proof. The statement follows by Theorems 3.19 and 2.8. $\square$

Acknowledgement. We thank the anonymous referee for his / her remarks, which have improved the final version of this paper.

REFERENCES

[1] M. A. ARIÑO AND B. MUCKENHOUPT, *Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions*, Trans. Amer. Math. Soc. **320** (1990), no. 2, 727–735, doi:10.2307/2001699, MR989570 (90k:42034).

[2] J. BASTERO, M. MILMAN AND F. J. RUIZ, *Rearrangement of Hardy-Littlewood maximal functions in Lorentz spaces*, Proc. Amer. Math. Soc. **128** (2000), no. 1, 65–74, doi:10.1090/S0002-9939-99-05128-X, MR1641637 (2000c:42020).

[3] C. BENNETT AND R. SHARPLEY, *Interpolation of operators*, Pure and Applied Mathematics, vol. 129, Academic Press, Inc., Boston, MA, 1988, MR928802 (89e:46001).

[4] G. BENNETT AND K.-G. GROSSE-ERDMANN, *Weighted Hardy inequalities for decreasing sequences and functions*, Math. Ann. **334** (2006), no. 3, 489–531, doi:10.1007/s00208-005-0678-7, MR2207873.

[5] D. W. BOYD, *The Hilbert transform on rearrangement-invariant spaces*, Canad. J. Math. **19** (1967), 599–616, MR0212512 (35 #3383).

[6] M. J. CARRO, A. GARCIA DEL AMO AND J. SORIA, *Weak-type weights and normable Lorentz spaces*, Proc. Amer. Math. Soc. **124** (1996), no. 3, 849–857, doi:10.1090/S0002-9939-96-03214-5, MR1307501.

[7] M. J. CARRO, A. GOGATISHVILI, J. MARTÍN AND L. PICK, *Weighted inequalities involving two Hardy operators with applications to embeddings of function spaces*, J. Operator Theory **59** (2008), no. 2, 309–332, MR2411048.

[8] M. J. CARRO AND ORTIZ-CARABALLO, *Boundedness of integral operators on decreasing functions*, Proc. Roy. Soc. Edinburgh Sect. A, **145** (2015), no. 4, 725–744, doi:10.1017/S0308210515000098, MR3371573.

[9] M. CARRO, L. PICK, J. SORIA AND V. D. STEPANOV, *On embeddings between classical Lorentz spaces*, Math. Inequal. Appl. **4** (2001), no. 3, 397–428, doi:10.7153/mia-04-37, MR1841071 (2002d:46026).

[10] M. J. CARRO, J. A. RAPOSO AND J. SORIA, *Recent developments in the theory of Lorentz spaces and weighted inequalities*, Mem. Amer. Math. Soc. **187** (2007), no. 877, xii+128, doi:10.1090/memo/0877, MR2308059 (2008b:42034).

[11] M. J. CARRO AND J. SORIA, *Boundedness of some integral operators*, Canad. J. Math. **45** (1993), no. 6, 1155–1166, doi:10.4153/CJM-1993-064-2, MR1247539.

[12] M. J. CARRO AND J. SORIA, *Weighted Lorentz spaces and the Hardy operator*, J. Funct. Anal. **112** (1993), no. 2, 480–494, doi:10.1006/jfan.1993.1042, MR1213148 (94f:42025).

[13] M. J. CARRO AND J. SORIA, *The Hardy-Littlewood maximal function and weighted Lorentz spaces*, J. London Math. Soc. (2) **55** (1997), no. 1, 146–158, doi:10.1112/S0024610796004462, MR1423292.

[14] K. M. CHONG AND N. M. RICE, *Equimeasurable rearrangements of functions*, Queen's University, Kingston, Ont., 1971, Queen's Papers in Pure and Applied Mathematics, no. 28, MR0372140 (51 #8357).

[15] A. CIANCHI, R. KERMAN, B. OPIC AND L. PICK, *A sharp rearrangement inequality for the fractional maximal operator*, Studia Math. **138** (2000), no. 3, 277–284, MR1758860 (2001h:42029).

[16] M. CWIKEL AND E. PUSTYLNIK, *Weak type interpolation near "endpoint" spaces*, J. Funct. Anal. **171** (2000), no. 2, 235–277, doi:10.1006/jfan.1999.3502, MR1745635 (2001b:46118).

[17] R. YA. DOKTORSKII, *Reiterative relations of the real interpolation method*, Dokl. Akad. Nauk SSSR **321** (1991), no. 2, 241–245 (Russian), English transl., Soviet Math. Dokl. **44**, (1992), no. 3, 665–669, MR1153547 (93b:46143).

[18] D. E. EDMUNDS AND B. OPIC, *Boundedness of fractional maximal operators between classical and weak-type Lorentz spaces*, Dissertationes Math. (Rozprawy Mat.) **410** (2002), 50, doi:10.4064/dm410-0-1, MR1952673 (2004c:42040).

[19] D. E. EDMUNDS AND B. OPIC, *Alternative characterisations of Lorentz-Karamata spaces*, Czechoslovak Math. J. **58** (133) (2008), no. 2, 517–540, doi:10.1007/s10587-008-0033-8, MR2411107 (2009c:46044).

[20] W. D. EVANS AND B. OPIC, *Real interpolation with logarithmic functors and reiteration*, Canad. J. Math. **52** (2000), no. 5, 920–960, doi:10.4153/CJM-2000-039-2, MR1782334 (2001i:46115).

[21] J. GARCIA-CUERVA AND J. L. RUBIO DE FRANCIA, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, vol. 116, North-Holland Publishing Co., Amsterdam, 1985, Notas de Matemática [Mathematical Notes], 104.

[22] A. GOGATISHVILI, M. JOHANSSON, C. A. OKPOTI AND L.-E. PERSSON, *Characterisation of embeddings in Lorentz spaces*, Bull. Austral. Math. Soc. **76** (2007), no. 1, 69–92, doi:10.1017/S0004972700039484, MR2343440.

[23] A. GOGATISHVILI AND R. CH. MUSTAFAYEV,, *Weighted iterated Hardy-type inequalities*, Math. Inequal. Appl. **20** (2017), no. 3, 683–728, doi:10.7153/mia-20-45, MR3653914.

[24] A. GOGATISHVILI AND R. CH. MUSTAFAYEV, *Iterated Hardy-type inequalities involving suprema*, Math. Inequal. Appl. **20** (2017), no. 4, 901–927, doi:doi:10.7153/mia-20-57.

[25] A. GOGATISHVILI, B. OPIC AND L. PICK, *Weighted inequalities for Hardy-type operators involving suprema*, Collect. Math. **57** (2006), no. 3, 227–255, MR2264321 (2007g:26019).

[26] A. GOGATISHVILI AND L. PICK, *A reduction theorem for supremum operators*, J. Comput. Appl. Math. **208** (2007), no. 1, 270–279, MR2347749 (2009a:26013).

[27] A. GOGATISHVILI AND V. D. STEPANOV, *Reduction theorems for weighted integral inequalities on the cone of monotone functions*, Uspekhi Mat. Nauk, **68** (2013), no. 4 (412), 3–68, (Russian, with Russian summary), English transl., Russian Math. Surveys **68** (2013), no. 4, 597–664, MR3154814.

[28] L. GRAFAKOS, *Classical Fourier analysis*, 2nd ed., Graduate Texts in Mathematics, vol. **249**, Springer, New York, 2008, MR2445437 (2011c:42001).

[29] L. GRAFAKOS, *Modern Fourier analysis*, 2nd ed., Graduate Texts in Mathematics, vol. 250, Springer, New York, 2009, MR2463316 (2011d:42001).

[30] M. DE GUZMÁN, *Differentiation of integrals in* R^n, Lecture Notes in Mathematics, vol. 481, Springer-Verlag, Berlin-New York, 1975, with appendices by Antonio Córdoba, and Robert Fefferman, and two by Roberto Moriyón.

[31] A. KAMIŃSKA AND L. MALIGRANDA, *Order convexity and concavity of Lorentz spaces* $\Lambda_{p,w}$, $0 < p < \infty$, Studia Math. **160** (2004), no. 3, 267–286, doi:10.4064/sm160-3-5, MR2033403 (2005e:46047).

[32] R. KERMAN AND L. PICK, *Optimal Sobolev imbeddings*, Forum Math. **18** (2006), no. 4, 535–570, doi:10.1515/FORUM.2006.028, MR2254384 (2007g:46052).

[33] A. KUFNER, L.-E. PERSSON AND N. SAMKO, *Weighted inequalities of Hardy type*, 2nd ed., World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017, doi:10.1142/10052, MR3676556.

[34] A. KUFNER, L. MALIGRANDA AND L.-E. PERSSON, *The Hardy inequality*, Vydavatelský Servis, Plzeň, 2007, About its history and some related results, MR2351524 (2008j:26001).

[35] M. A. LECKBAND AND C. J. NEUGEBAUER, *Weighted iterates and variants of the Hardy-Littlewood maximal operator*, Trans. Amer. Math. Soc. **279** (1983), no. 1, 51–61, doi:10.2307/1999370, MR704601 (85c:42021).

[36] A. K. LERNER, *A new approach to rearrangements of maximal operators*, Bull. London Math. Soc. **37** (2005), no. 5, 771–777, doi:10.1112/S0024609305004698, MR2164840 (2006d:42032).

[37] J. LINDENSTRAUSS AND L. TZAFRIRI, *Classical Banach spaces. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], vol. 97, Springer-Verlag, Berlin-New York, 1979, Function spaces, MR540367 (81c:46001).

[38] M. MASTYŁO AND C. PÉREZ, *The Hardy-Littlewood maximal type operators between Banach function spaces*, Indiana Univ. Math. J. **61** (2012), no. 3, 883–900, doi:10.1512/iumj.2012.61.4708, MR3071689.

[39] C. J. NEUGEBAUER, *Iterations of Hardy-Littlewood maximal functions*, Proc. Amer. Math. Soc. **101** (1987), no. 2, 272–276, doi:10.2307/2045994, MR902540 (88k:42014).

[40] B. OPIC, *On boundedness of fractional maximal operators between classical Lorentz spaces*, Function spaces, differential operators and nonlinear analysis, (Pudasjärvi, 1999), Acad. Sci. Czech Repub., Prague, (2000), 187–196, MR1755309 (2001g:42043).

[41] B. OPIC AND A. KUFNER, *Hardy-type inequalities*, Pitman Research Notes in Mathematics Series, vol. 219, Longman Scientific & Technical, Harlow, 1990, MR1069756 (92b:26028).

[42] B. OPIC AND W. TREBELS, *Bessel potentials with logarithmic components and Sobolev-type embeddings*, Anal. Math. **26** (2000), no. 4, 299–319, doi:10.1023/A:1005691512014, (English, with English and Russian summaries), MR1805506 (2002b:46057).

[43] B. OPIC AND W. TREBELS, *Sharp embeddings of Bessel potential spaces with logarithmic smoothness*, Math. Proc. Cambridge Philos. Soc. **134** (2003), no. 2, 347–384, doi:10.1017/S0305004102006321, MR1972143 (2004c:46057).
[44] C. PÉREZ, *On sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator between weighted L^p-spaces with different weights*, Proc. London Math. Soc. (3) **71** (1995), no. 1, 135–157, doi:10.1112/plms/s3-71.1.135, MR1327936 (96k:42023).
[45] L. PICK, *Supremum operators and optimal Sobolev inequalities*, Function spaces, differential operators and nonlinear analysis (Pudasjärvi, 1999), Acad. Sci. Czech Repub., Prague, 2000, 207–219, MR1755311 (2000m:46075).
[46] L. PICK, *Optimal Sobolev embeddings – old and new*, Function spaces, interpolation theory and related topics (Lund, 2000), de Gruyter, Berlin, 2002, 403–411, MR1943297 (2003j:46054).
[47] E. PUSTYLNIK, *Optimal interpolation in spaces of Lorentz-Zygmund type*, J. Anal. Math. **79** (1999), 113–157, doi:10.1007/BF02788238, MR1749309 (2001a:46028).
[48] E. SAWYER, *Boundedness of classical operators on classical Lorentz spaces*, Studia Math. **96** (1990), no. 2, 145–158, MR1052631 (91d:26026).
[49] J. SORIA, *Lorentz spaces of weak-type*, Quart. J. Math. Oxford Ser. (2) **49**, (1998), no. 193, 93–103, doi:10.1093/qjmath/49.193.93, MR1617343 (99e:46037).
[50] E. M. STEIN, *Editor's note: the differentiability of functions in* $\mathbf{R}^n$, Ann. of Math. (2) **113** (1981), no. 2, 383–385, MR607898 (84j:35077).
[51] E. M. STEIN, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, no. 30, Princeton University Press, Princeton, N. J., (1970), MR0290095 (44 #7280).
[52] E. M. STEIN, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III, MR1232192 (95c:42002).
[53] V. D. STEPANOV, *Integral operators on the cone of monotone functions*, J. London Math. Soc. (2) **48** (1993), no. 3, 465–487, doi:10.1112/jlms/s2-48.3.465, MR1241782.
[54] A. TORCHINSKY, *Real-variable methods in harmonic analysis*, Pure and Applied Mathematics **123**, Academic Press, Inc., Orlando, FL, 1986, MR869816 (88e:42001).

(Received November 16, 2016)

Rza Mustafayev
Institute of Mathematics and Mechanics
Academy of Sciences of Azerbaijan
B. Vahabzade St. 9, Baku, AZ 1141, Azerbaijan
and
Department of Mathematics
Faculty of Science
Karamanoglu Mehmetbey University
Karaman, 70100, Turkey
e-mail: rzamustafayev@gmail.com

Nevin Bilgiçli
Department of Mathematics
Faculty of Science and Arts, Kirikkale University
71450 Yahsihan, Kirikkale, Turkey
e-mail: nevinbilgicli@gmail.com