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Properties and Inference for a New Class of Generalized Rayleigh Distributions with an Application

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Abstract: In the present paper, we introduce a new form of generalized Rayleigh distribution called the Alpha Power generalized Rayleigh (APGR) distribution by following the idea of extension of the distribution families with the Alpha Power transformation. The introduced distribution has the more general form than both the Rayleigh and generalized Rayleigh distributions and provides a better fit than the Rayleigh and generalized Rayleigh distributions for more various forms of the data sets. In the paper, we also obtain explicit forms of some important statistical characteristics of the APGR distribution such as hazard function, survival function, mode, moments, characteristic function, Shannon and Rényi entropies, stress-strength probability, Lorenz and Bonferroni curves and order statistics. The statistical inference problem for the APGR distribution is investigated by using the maximum likelihood and least-square methods. The estimation performances of the obtained estimators are compared based on the bias and mean square error criteria by a conducted Monte-Carlo simulation on small, moderate and large sample sizes. Finally, a real data analysis is given to show how the proposed model works in practice.

Keywords: Alpha power transformation; Maximum likelihood estimate; Least-square estimate; Shannon entropy; Generalized Rayleigh distribution.

MSC: 62E20,62F10

1 Introduction

The famous distribution families have been successfully used in modeling real-world data sets, until recently. However, it is well known that the performances of these distributions in the modeling of complex real-world data sets are not always at the desired level. In recent years, a number of researchers who take into account this situation have focused on introducing the flexible distribution families in order to the modeling of data sets in a wide variety of complex structures and have made several breakthroughs by giving various continuous distribution generating methods, especially in lifetime distributions. These distribution produce methods lay out a new distribution taking a baseline distribution. The baseline distributions are always a special case of the newly obtained distribution. Hence, the produced distribution has the characteristics of the baseline distribution and provides better data fit than the baseline distribution. There are numerous papers in the literature that create a new distribution using a baseline distribution and draw attention to its advantages. We refer readers to [1–5] for further information on generating a new distribution family by using a baseline distribution.

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The Rayleigh distribution, which has only a shape parameter, was originally introduced by a study of Rayleigh on a problem of acoustic. The distribution has a strong modeling ability of positive valued and skewed data obtained from many fields such as engineering, biology, life sciences, reliability and etc. The Rayleigh distribution is a distribution related to Gamma, Weibull, Exponential and Rice distributions. However, it has a disadvantage since the distribution has only a single shape parameter in which plays a crucial role in describing the various behaviors of the distribution. Fortunately, to overcome this disadvantage of Rayleigh distribution, there are many generalizations of the distribution such as generalized Rayleigh distribution [6], transmuted Rayleigh distribution [7], Weibull Rayleigh distribution [8], inverted exponentiated Rayleigh distribution [9] and the slashed exponentiated Rayleigh distribution [10]. The generalized Rayleigh distribution is the most widely used among these generalizations. The generalized Rayleigh distribution has also some important generalizations recently introduced to achieve optimal data fit, such as the Kumaraswamy generalized Rayleigh [11], the beta generalized Rayleigh [12], the slashed generalized Rayleigh [13] and the Marshall-Olkin extended generalized Rayleigh [14]. In the literature, there are also many published papers on the estimation of the parameters of Rayleigh and generalized Rayleigh distributions for the various data types, see [15–24].

The main motivation of this paper is to introduce a more flexible lifetime distribution than the Rayleigh and generalized Rayleigh distribution to be used for the modeling of data sets in wide variety structures. In the aim of this context, in the study, a new three-parameter family of Rayleigh distribution which is named alpha power generalized Rayleigh distribution ($APGR$) is derived using the alpha power transform (APT) method recently introduced by Mahdavi and Kundu [5]. Both Rayleigh and generalized Rayleigh distributions are the special cases of $APGR$ distribution. Therefore, APGR distribution has more data modeling capability than the Rayleigh and generalized Rayleigh. Further, the $APGR$ distribution is an important alternative to famous distributions like Gamma, Weibull, and exponential for modeling the data observed from industrial and physical phenomena.

The rest of the paper is organized as follows. In section 2, we introduce the $APGR$ distribution. We discuss some important statistical characteristics of the $APGR$ distribution in section 3. In section 4, statistical inference problem for the $APGR$ distribution is investigated according to maximum likelihood (ML) and least-square (LSq) methods. Section 5 includes a comprehensive Monte-Carlo simulation study display the estimation performance of the estimators derived in section 4. A real-world data set is analyzed in section 6 for illustrative purposes. Finally, section 7 concludes the paper.

2 Definition and properties of the $APGR$ Distribution

In this section, we derive the probability density function (pdf) and cumulative distribution function (cdf) of the $APGR$ distribution by using the APT method given in [5] and study some distributional properties of the $APGR$ distribution. Before progressing for further, we recall the generalized Rayleigh distribution. The pdf of the generalized Rayleigh distribution is

$$g(x;\beta,\lambda) = 2\beta\lambda^2 x e^{-(\lambda x)^2}\left(1 - e^{-(\lambda x)^2}\right)^{\beta-1}, \quad x > 0, \tag{1}$$

and its cdf is

$$G(x,\beta,\lambda) = \left(1 - e^{-(\lambda x)^2}\right)^{\beta}, \quad x > 0, \tag{2}$$

where β and λ is the positive and real valued scale parameter and shape parameters of the distribution, respectively. Generalized Rayleigh distribution was originally studied by Surles and Padgett [6] as the two-parameter Burr Type X distribution. Then, the distribution was called the generalized Rayleigh distribution by Raqab and Kundu [25].

Now, we introduce the $APGR$ distribution by using generalized Rayleigh distribution as a baseline distribution in the APT method.

Definition 1. A random variable X is said to have a $APGR$ distribution with parameters α, β and λ, if it has the following pdf and cdf

$$f_{APGR}(x;\alpha,\beta,\lambda)=\begin{cases}\frac{\ln\alpha}{\alpha-1}2\beta\lambda^2xe^{-(\lambda x)^2}\left(1-e^{-(\lambda x)^2}\right)^{\beta-1}\alpha^{\left(1-e^{-(\lambda x)^2}\right)^{\beta}}, & x>0,\ \alpha>0\wedge\alpha\neq1\\ 2\beta\lambda^2xe^{-(\lambda x)^2}\left(1-e^{-(\lambda x)^2}\right)^{\beta-1}, & x>0,\ \alpha=1\\ 0 & otherwise\end{cases} \quad (3)$$

and

$$F_{APGR}(x;\alpha,\beta,\lambda)=\begin{cases}\frac{\alpha^{\left(1-e^{-(\lambda x)^2}\right)^{\beta}}-1}{\alpha-1}, & x>0,\ \alpha>0\wedge\alpha\neq1\\ \left(1-e^{-(\lambda x)^2}\right)^{\beta}, & x>0,\ \alpha=0\end{cases}, \quad (4)$$

respectively.

Considering the cdf given by equation (4), the survival and hazard functions of the $APGR$ distribution can be easily written as in the following forms:

$$S_{APGR}(x;\alpha,\beta,\lambda)=\begin{cases}\frac{\alpha-\alpha^{\left(1-e^{-x^2\lambda^2}\right)^{\beta}}}{\alpha-1}, & x>0,\ \alpha>0\wedge\alpha\neq1\\ 1-\left(1-e^{-(\lambda x)^2}\right)^{\beta}, & x>0,\ \alpha=0\end{cases} \quad (5)$$

and

$$h_{APGR}(x;\alpha,\beta,\lambda)=\begin{cases}2\beta\lambda^2xe^{-(\lambda x)^2}\left(1-e^{-(\lambda x)^2}\right)^{\beta-1}\frac{\alpha^{\left(\left(1-e^{-(\lambda x)^2}\right)^{\beta}-1\right)}}{1-\alpha^{\left(\left(1-e^{-(\lambda x)^2}\right)^{\beta}-1\right)}}\ln\alpha & x>0,\ \alpha>0\wedge\alpha\neq1\\ \frac{2\beta\lambda^2xe^{-(\lambda x)^2}\left(1-e^{-(\lambda x)^2}\right)^{\beta-1}}{1-\left(1-e^{-(\lambda x)^2}\right)^{\beta}} & x>0,\ \alpha=0\end{cases}. \quad (6)$$

From now on, a random variable X distributed the $APGR$ with parameters α, β and λ will be indicated as $X\sim APGR\,(\alpha,\beta,\lambda)$. By considering the equation (11) in [5], the p-th quantile of the $APGR$ distribution, say Q_p, is immediately obtained as below

$$Q_p=\frac{\left(-\ln\left(1-\left(\frac{\ln\left(-\frac{\alpha}{1-\alpha}+\frac{1}{1-\alpha}+\alpha p-p\right)}{\ln(\alpha)}\right)^{1/\beta}\right)\right)^{1/2}}{\lambda}. \quad (7)$$

Thus, when $\alpha\neq1$, the median of the $APGR$ distribution is obtained as

$$M=Q_{0.5}=\frac{1}{\lambda}\left(-\ln\left(1-\left(\frac{\ln\left(\frac{\alpha+1}{2}\right)}{\ln\alpha}\right)^{\frac{1}{\beta}}\right)\right)^{1/2} \quad (8)$$

and when $\alpha=1$, the median of the $APGR$ distribution is equal to median of the generalized Rayleigh distribution.

Now, we discuss the shape behavior of the pdf $f_{APGR}(x;\alpha,\beta,\lambda)$. When X tends to 0 and X tends to ∞, the pdf $f_{APGR}(x;\alpha,\beta,\lambda)$ comply with the following behaviors

$$\lim_{x\to0^+}f_{APGR}(x;\alpha,\beta,\lambda)=0$$

and

$$\lim_{x\to\infty}f_{APGR}(x;\alpha,\beta,\lambda)=0,$$

respectively.

Theorem 1. The $APGR$ distribution is unimodal.

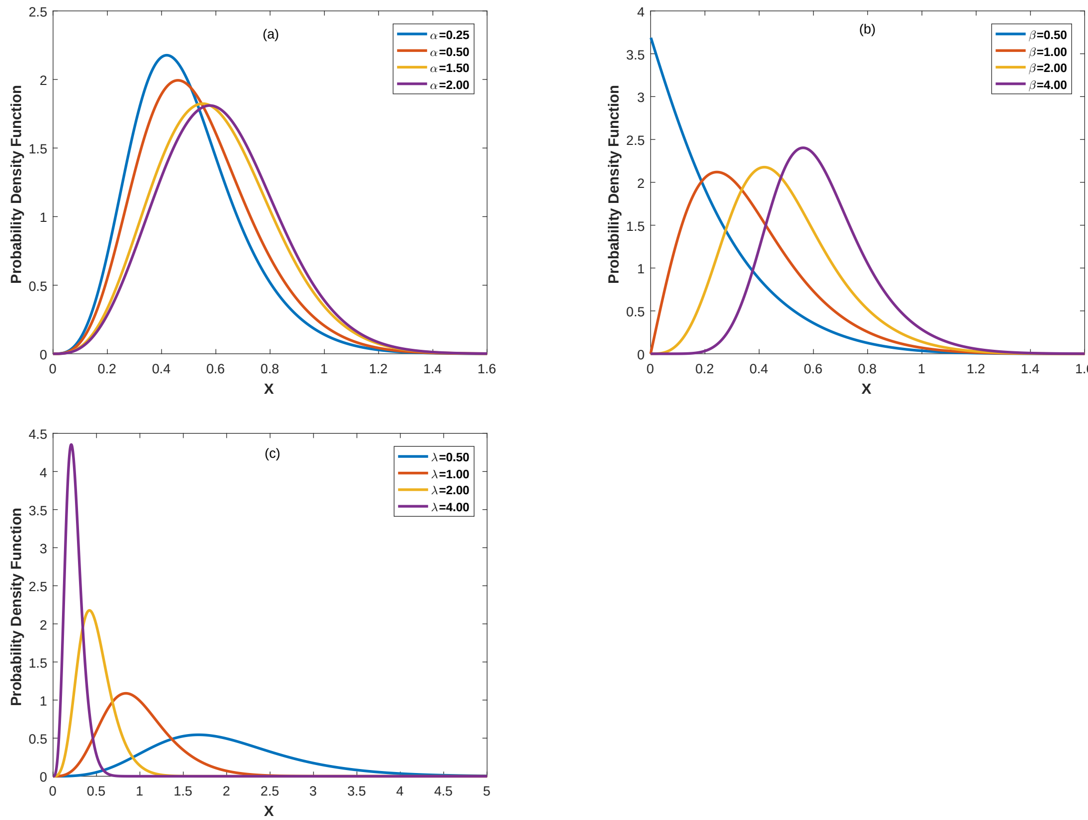

Figure 1: The pdf of the *APGR* distribution, when (**a**): $\alpha = 0.25, 0.5, 1.5, 2.$, $\beta = 2$ and $\lambda = 2$; (**b**): $\alpha = 0.25$, $\beta = 0.5, 1, 2, 4$ and $\lambda = 2$, (**c**): $\alpha = 0.25$, $\beta = 2$ and $\lambda = 0.5, 1, 2, 4$

Proof. First derivative of the pdf $f_{APGR}(x; \alpha, \beta, \lambda)$ given by equation (3) is

$$f'_{APGR}(x; \alpha, \beta) = \begin{cases} \dfrac{\beta^2 \ln(\alpha) e^{-2\beta x} \alpha^{1-\frac{e^{-\beta x}(\beta x+\beta+1)}{\beta+1}} \left(\beta^2 (x+1)^2 \ln(\alpha) - (\beta+1) e^{\beta x}(\beta x+\beta-1)\right)}{(\alpha-1)(\beta+1)^2} & , \quad \alpha \neq 1 \\ \dfrac{\beta^2 e^{-\beta x} (1-\beta-\beta x)}{\beta+1} & , \quad \alpha = 1 \end{cases} . \tag{9}$$

When $\alpha = 1$, that is the distribution is a generalized Rayleigh, mode of the distribution can be easily obtained from solution of the equation

$$\frac{\beta^2 e^{-\beta x} (1 - \beta - \beta x)}{\beta + 1} = 0. \tag{10}$$

When $\alpha \neq 1$, the derivative $f'_{APGR}(x; \alpha, \beta, \lambda)$ is a strictly decreasing and continuous function of x and $\lim_{x \to 0^+} f'_{APGR}(x; \alpha, \beta, \lambda)$ is positive and $f'_{APGR}(x; \alpha, \beta, \lambda)$ takes negative values as $x \to \infty$. Thus, we can say the $f'_{APGR}(x; \alpha, \beta, \lambda)$ has only one zero according to intermediate value theorem and the pdf $f_{APGR}(x; \alpha, \beta, \lambda)$ is unimodal. □

We present a figure to show the shape behavior of the *APGR* distribution for illustrative purposes. Fig.1 a,b,c display the some of the possible shapes of the pdf of the *APGR* distribution for different values of the parameters α, β and λ.

3 Some Important Characteristics of the APGR Distribution

In this section, the moments, moment generating function and related measures such as mean, variance, skewness and kurtosis are obtained for the $APGR$ distribution. In addition, the distribution of order statistics, stress-strength probability and Shannon and Rényi entropies and the Lorenz and Bonferroni curves of the $APGR$ distribution are also obtained in this section.

Let we first introduce the Lemma 1 to obtain the moments of $APGR$ distribution.

Lemma 1. Let X be a random variable with pdf given by equation (3). For any real numbers $a > 0, b > 0, L > 0, r \geq 0$ and $\delta \geq 0$, the integral

$$\xi(a,b,L,r,\delta) = \int_0^\infty x^{r+1} e^{-(Lx)^2} \left(1 - e^{-(\lambda x)^2}\right)^{b-1} a^{\left(1-e^{-(Lx)^2}\right)^b} e^{\delta x} dx \tag{11}$$

is calculated as

$$\begin{aligned}\xi(a,b,L,r,\delta) = \sum_{i=0}^{\infty}\sum_{j=0}^{ib}\sum_{k=0}^{b-1} \Bigg\{ \frac{(\log a)^i}{i!} (-1)^{j+k} \binom{ib}{j}\binom{b-1}{k} \frac{1}{2} \left(L^2(j+k+1)\right)^{\frac{1}{2}(-r-3)} \times \\ \left(\delta\Gamma\left(\frac{r+3}{2}\right) {}_1F_1\left(\frac{r+3}{2};\frac{3}{2};\frac{\delta^2}{4(j+k+1)L^2}\right) + \right. \\ \left.\Gamma\left(\frac{r}{2}+1\right)\sqrt{L^2(j+k+1)}\, {}_1F_1\left(\frac{r+2}{2};\frac{1}{2};\frac{\delta^2}{4(j+k+1)L^2}\right)\right)\Bigg\}\end{aligned} \tag{12}$$

where ${}_1F_1(.;.;.)$ is indicate the hypergeometric function, see [27]

Proof. See Appendix A for proof of Lemma 1. □

Obviously, by using the Lemma 1, the r-th moment, moment generating function, characteristic function, mean and variance of the $APGR$ distribution are easily obtained as

$$\mu_r' = E\left(X^r\right) = \frac{\ln\alpha}{\alpha-1} 2\beta\lambda^2 \xi(\alpha,\beta,\lambda,r,0), \tag{13}$$

$$M_X(t) = E\left(e^{tx}\right) = \frac{\ln\alpha}{\alpha-1} 2\beta\lambda^2 \xi(\alpha,\beta,\lambda,0,t), \tag{14}$$

$$\Phi_X(t) = E\left(e^{itx}\right) = \frac{\ln\alpha}{\alpha-1} 2\beta\lambda^2 \xi(\alpha,\beta,\lambda,0,it) \tag{15}$$

$$\mu = \mu_1' = E(X) = \frac{\ln\alpha}{\alpha-1} 2\beta\lambda^2 \xi(\alpha,\beta,\lambda,1,0), \tag{16}$$

and $\sigma^2 = Var(X) = \mu_2' - \left(\mu_1'\right)^2$,respectively.

Now, we derive the central moments and cumulants of the $APGR$ distribution. By using the raw moments given in equation (13), r-th central moment of the $APGR$ distribution is obtained as follow

$$\begin{aligned}\mu_r &= \sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}\mu_j'\left(\mu_1'\right)^{r-j} \\ &= \sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}\frac{\ln\alpha}{\alpha-1}2\beta\lambda^2\xi(\alpha,\beta,\lambda,j,0)\left(\frac{\ln\alpha}{\alpha-1}2\beta\lambda^2\xi(\alpha,\beta,\lambda,1,0)\right)^{r-j}.\end{aligned} \tag{17}$$

Therefore, using the central moments given by equation (17), the second, third and fourth cumulants κ_2, κ_3 and κ_4 can be expressed as $\kappa_2 = \mu_2$, $\kappa_3 = \mu_3$ and $\kappa_4 = \mu_4 - 3\mu_2^2$, respectively. The skewness and the kurtosis coefficients of the $APGR$ distribution are calculated by $\gamma_1 = \kappa_3/\kappa_2^{3/2}$and $\gamma_2 = \kappa_4/\kappa_2^2$.

3.1 Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample from $APGR(\alpha, \beta, \lambda)$ distribution and $X_{(1)}, X_{(2)}, \ldots X_{(n)}$ denote their order statistics. The pdf of the random variable $X_{(r)}$, $(r = 1, 2, \ldots, n)$ is obtained as

$$\begin{aligned}
f_{X_{(r)}}(x) &= \frac{n!}{(r-1)!(n-r)!} F_{APGR}(x,\alpha,\beta,\lambda)^{r-1} f_{APGR}(x,\alpha,\beta,\lambda)(1-F_{APGR}(x,\alpha,\beta,\lambda))^{n-r} \\
&= \left\{ \frac{n!}{(r-1)!(n-r)!} \frac{\ln\alpha}{\alpha-1} 2\beta\lambda^2 x e^{-(\lambda x)^2} \left(\frac{\alpha^{\left(1-e^{-(\lambda x)^2}\right)^\beta} - 1}{\alpha-1} \right)^{r-1} \right. \\
&\quad \left. \left(1 - \frac{\alpha^{\left(1-e^{-(\lambda x)^2}\right)^\beta} - 1}{\alpha-1} \right)^{n-r} \left(1-e^{-(\lambda x)^2}\right)^{\beta-1} \alpha^{\left(1-e^{-(\lambda x)^2}\right)^\beta} \right\} \\
&= \frac{n!}{(r-1)!(n-r)!} 2\beta\lambda^2 x e^{-(\lambda x)^2} \frac{\ln\alpha}{\alpha-1} \left(\frac{\alpha^\zeta-1}{\alpha-1}\right)^{r-1} \left(1-\frac{\alpha^\zeta-1}{\alpha-1}\right)^{n-r} \left(\zeta^{1-\frac{1}{\beta}}\alpha^\zeta\right)
\end{aligned} \tag{18}$$

where, $\zeta = \left(1-e^{-(\lambda x)^2}\right)^\beta$. In particular, pdf of the first and n-th order statistics can be easily derived from equation (18) as

$$f_{X_{(1)}}(x) = n2\beta\lambda^2 x e^{-(\lambda x)^2} \frac{\ln\alpha}{\alpha-1} \left(1-\frac{\alpha^\zeta-1}{\alpha-1}\right)^{n-1} \left(\zeta^{1-\frac{1}{\beta}}\alpha^\zeta\right) \tag{19}$$

and

$$f_{X_{(n)}}(x) = n2\beta\lambda^2 x e^{-(\lambda x)^2} \frac{\ln\alpha}{\alpha-1} \left(\frac{\alpha^\zeta-1}{\alpha-1}\right)^{n-1} \left(\zeta^{1-\frac{1}{\beta}}\alpha^\zeta\right), \tag{20}$$

respectively.

3.2 Stress-strength probability

We suppose that X and Y be random variables from $APGR(\alpha_1, \beta_1, \lambda_1)$ and $Y \sim APGR(\alpha_2, \beta_2, \lambda_2)$ distributions, respectively. In this situation, the stress-strength probability is calculated by $R = P(Y < X)$, where Y represents the 'stress' and X represents the 'strength' to sustain the stress. For $APGR$ distribution, stress-strength probability $P(Y < X)$ is obtained as below

$$\begin{aligned}
R = P(Y<X) &= \int_0^\infty P(Y<X \mid X=x) f_X(x)\,dx = \int_0^\infty f_X(x) F_Y(x)\,dx \\
&= \int_0^\infty \frac{\ln\alpha_1}{\alpha_1-1} 2\beta_1\lambda_1^2 x e^{-(\lambda_1 x)^2} \left(1-e^{-(\lambda_1 x)^2}\right)^{\beta_1-1} \alpha_1^{\left(1-e^{-(\lambda_1 x)^2}\right)^{\beta_1}} F_Y(x)\,dx \\
&= \int_0^\infty \frac{\ln\alpha_1}{\alpha_1-1} 2\beta_1\lambda_1^2 x e^{-(\lambda_1 x)^2} \left(1-e^{-(\lambda_1 x)^2}\right)^{\beta_1-1} \alpha^{\left(1-e^{-(\lambda_1 x)^2}\right)^{\beta_1}} \frac{\alpha_2^{\left(1-e^{-(\lambda_2 x)^2}\right)^{\beta_2}} - 1}{\alpha_2-1} dx \\
&= \frac{1}{(\alpha_2-1)} \left\{ \frac{2\beta_1\lambda_1^2 \ln\alpha_1}{(\alpha_1-1)} \int_0^\infty x e^{-(\lambda_1 x)^2} \left(1-e^{-(\lambda_1 x)^2}\right)^{\beta_1-1} \alpha_1^{\left(1-e^{-(\lambda x)^2}\right)^\beta} \alpha_2^{\left(1-e^{-(\lambda_2 x)^2}\right)^{\beta_2}} dx - E[X] \right\}.
\end{aligned} \tag{21}$$

Further, using the Lemma 1, we have

$$R = \frac{1}{(\alpha_2-1)} \Lambda(\alpha_2,\beta_2,\lambda_2) - \frac{2\beta_1\lambda_1^2 \ln\alpha_1}{(\alpha-1)(\alpha_2-1)} \xi(\alpha_1,\beta_1,\lambda_1,1,0), \tag{22}$$

where $\Lambda(\alpha_2,\beta_2,\lambda_2) = E\left[\alpha_2^{\left(1-e^{-(\lambda_2 x)^2}\right)^{\beta_2}}\right]$.

3.3 Shannon and Rényi Entropies

The entropy is quite important as a measure of variation or uncertainty of a random variable. In this section, we investigate the Shannon and Rényi entropies of the $APGR$ distribution. The Shannon entropy of a random variable X with pdf $f(x)$ is defined as, see [26],

$$\mathcal{H}(X) = E[-\ln f(x)]. \tag{23}$$

Hence, the Shannon entropy of $APGR(\alpha, \beta, \lambda)$ distribution is obtained as

$$\begin{aligned}
\mathcal{H}(X) &= -\int_0^\infty f_{APGR}(x, \alpha, \beta, \lambda) \ln(f_{APGR}(x, \alpha, \beta, \lambda))\, dx \\
&= -\int_0^\infty f_{APGR}(x, \alpha, \beta, \lambda) [\ln(\ln \alpha) - \ln(\alpha - 1) + \ln 2 + \ln \beta + 2 \ln \lambda + \ln x - \\
&\quad \lambda^2 x^2 + (\beta - 1) \ln\left(1 - e^{-(\lambda x)^2}\right) + \ln \alpha \left(1 - e^{-(\lambda x)^2}\right)^\beta \Big] dx \\
&= -\{\ln(\ln \alpha) - \ln(\alpha - 1) + \ln 2 + \ln \beta + 2 \ln \lambda + E[\ln(x)] - \\
&\quad \lambda^2 E\left[x^2\right] + (\beta - 1) E\left[\ln\left(1 - e^{-(\lambda x)^2}\right)\right] + \ln \alpha E\left[\left(1 - e^{-(\lambda x)^2}\right)^\beta\right]\Big\}.
\end{aligned} \tag{24}$$

By applying the Lemma 1 to equation (24), the Shannon entropy $\mathcal{H}(X)$ is written as

$$\begin{aligned}
\mathcal{H}(X) = -\{&\ln(\ln \alpha) - \ln(\alpha - 1) + \ln 2 + \ln \beta + 2 \ln \lambda + Y_X - \\
&\frac{\ln \alpha}{\alpha - 1} 2\beta\lambda^4 \xi(\alpha, \beta, \lambda, 2, 0) + (\beta - 1) \vartheta_X + \ln \alpha \varsigma_X\Big\},
\end{aligned} \tag{25}$$

where $Y_X = E(\ln(X))$, $\vartheta_X = E\left[\ln\left(1 - e^{-(\lambda x)^2}\right)\right]$ and $\varsigma_X = E\left[\left(1 - e^{-(\lambda x)^2}\right)^\beta\right]$ and these expectations can be easily calculated numerically.

Now, we calculate the Rényi entropy of the $APGR$ distribution. We first recall the definition of the Rényi entropy. The Rényi entropy of a random variable X with pdf f is given by

$$RE_X(\xi) = \frac{1}{1 - \xi} \ln\left\{\int_{-\infty}^\infty [f(x)]^\xi\, dx\right\}. \tag{26}$$

By using the pdf (3) in the equation (26), Rényi entropy of the $APGR$ distribution is obtained as

$$RE_X(\xi) = \frac{1}{1 - \xi} \ln\left(\left(\frac{2\beta\lambda^2 \ln \alpha}{\alpha - 1}\right)^\xi \sum_{i=0}^\infty \frac{(\ln \alpha)^i}{i!} \sum_{j=0}^{i\beta+\beta-1} \binom{i\beta + \beta - 1}{j} (-1)^j \frac{1}{2} \Gamma\left(\frac{\xi + 1}{2}\right) \left((j + 1)\lambda^2 \xi\right)^{\frac{1}{2}(-\xi-1)}\right), \tag{27}$$

see Appendix B for calculation of the Rényi entropy of the $APGR$ distribution.

3.4 Lorenz and Bonferroni Curves

Lorenz and Bonferroni curves are two graphical representations to the measure inequality of distribution of a random variable. The Lorenz and Bonferroni curves for a random variable X are defined as the plot of

$$L(p) = \frac{1}{\mu} \int_0^q x f(x)\, dx, \tag{28}$$

and

$$B(p) = \frac{1}{p\mu}\int_0^q xf(x)\,dx, \tag{29}$$

respectively, against $F(x)$, where μ is indicate the expectation of the random variable X and $q = F^{-1}(p)$ also $L(p)$ and $B(p)$ are called the Lorenz index and Bonferroni index, respectively. If the expectation (16) and pdf (3) are used in the equation (28), the Lorenz index of $APGR$ distribution is obtained as

$$\begin{aligned} L(p) &= \frac{1}{\frac{\ln\alpha}{\alpha-1}2\beta\lambda^2\xi(\alpha,\beta,\lambda,1,0)}\int_0^q xf_{APGR}(x)\,dx \\ &= \frac{1}{\frac{\ln\alpha}{\alpha-1}2\beta\lambda^2\xi(\alpha,\beta,\lambda,1,0)}\int_0^q x\left(\frac{\ln\alpha}{\alpha-1}2\beta\lambda^2 xe^{-(\lambda x)^2}\left(1-e^{-(\lambda x)^2}\right)^{\beta-1}\alpha^{\left(1-e^{-(\lambda x)^2}\right)^{\beta}}\right)dx \\ &= \frac{1}{\xi(\alpha,\beta,\lambda,1,0)}\int_0^q x\left(\frac{\ln\alpha}{\alpha-1}2\beta\lambda^2 xe^{-(\lambda x)^2}\left(1-e^{-(\lambda x)^2}\right)^{\beta-1}\alpha^{\left(1-e^{-(\lambda x)^2}\right)^{\beta}}\right)dx \end{aligned} \tag{30}$$

Following steps of the proof of Lemma 1, the Lorenz index (30) is immediately written as

$$\begin{aligned} L(p) = \frac{1}{\xi(\alpha,\beta,\lambda,1,0)}&\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{i\beta+\beta-1}\binom{i\beta+\beta-1}{j}(-1)^k\times\right. \\ &\left.\frac{(\ln\alpha)^i}{\alpha-1}\frac{\sqrt{\pi}\operatorname{erf}\left(\sqrt{j+1}\lambda q\right)-2\sqrt{j+1}\lambda qe^{-(j+1)\lambda^2q^2}}{4(j+1)^{3/2}\lambda^3}\right\}, \end{aligned} \tag{31}$$

where erf(.) is indicate the error function, see [27]. Similarly, the Bonferroni index of $APGR$ distribution is also obtained as

$$\begin{aligned} B(p) = \frac{1}{p\xi(\alpha,\beta,\lambda,1,0)}&\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{i\beta+\beta-1}\binom{i\beta+\beta-1}{j}(-1)^k\times\right. \\ &\left.\frac{(\ln\alpha)^i}{\alpha-1}\frac{\sqrt{\pi}\operatorname{erf}\left(\sqrt{j+1}\lambda q\right)-2\sqrt{j+1}\lambda qe^{-(j+1)\lambda^2q^2}}{4(j+1)^{3/2}\lambda^3}\right\}, \end{aligned} \tag{32}$$

4 Inference

In this section, we consider the statistical inference problem for $APGR(\alpha,\beta,\lambda)$ distribution. We employ the ML and LSq methods to obtaining the estimators of the unknown parameters α, β, and λ.

4.1 ML estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample from $APGR(\alpha,\beta,\lambda)$ distribution. The log-likelihood function of the random variables X_i, $i = 1, 2, \ldots, n$ can be easily written from equation (3) as

$$\begin{aligned} L(\alpha,\beta,\lambda;\ X_1,X_2,\ldots,X_n) &= n(\ln(\ln\alpha)-\ln(\alpha-1)+\ln 2+\ln\beta+2\ln\lambda)+\sum_{i=1}^{n}\ln x_i - \\ &\lambda^2\sum_{i=1}^{n}x_i^2+(\beta-1)\sum_{i=1}^{n}\ln\left(1-e^{-(\lambda x_i)^2}\right)+\ln\alpha\sum_{i=1}^{n}\left(1-e^{-(\lambda x_i)^2}\right)^{\beta}. \end{aligned} \tag{33}$$

Thus, by derivating the log-likelihood function given in equation (33) with respect to parameters α, β and λ, we can write the following likelihood equations

$$\frac{\partial L}{\partial \alpha} = -\frac{n}{\alpha(\ln\alpha)(\alpha-1)}(\alpha\ln\alpha-\alpha+1)+\frac{1}{\alpha}\sum_{i=1}^{n}\left(1-e^{-(\lambda x_i)^2}\right)^{\beta}=0, \tag{34}$$

$$\frac{\partial L}{\partial \beta} = \frac{n}{\beta}+\sum_{i=1}^{n}\ln\left(1-e^{-(\lambda x_i)^2}\right)+\beta\ln\alpha\sum_{i=1}^{n}\left(1-e^{-(\lambda x_i)^2}\right)^{\beta-1}=0 \tag{35}$$

and

$$\frac{\partial L}{\partial \lambda} = \frac{2n}{\lambda}-2\lambda\sum_{i=1}^{n}x_i^2-2\lambda(\beta-1)\sum_{i=1}^{n}x_i^2\frac{e^{-\lambda^2x_i^2}}{e^{-\lambda^2x_i^2}-1}-2\beta\lambda\ln\alpha\sum_{i=1}^{n}x_i^2\frac{e^{-x_i^2\lambda^2}}{e^{-x_i^2\lambda^2}-1}\left(1-e^{-x_i^2\lambda^2}\right)^{\beta}=0 \tag{36}$$

Unfortunately, the ML estimators of the parameters α, β and λ cannot be explicitly derived from equations (34), (35) and (36). However, we can obtain the ML estimates of the parameters α, β and λ, say $\hat{\alpha}_{ML}$, $\hat{\beta}_{ML}$ and $\hat{\lambda}_{ML}$, respectively, from the simultaneous numerical solution of equations (34), (35) and (36).

4.2 LSq Estimation

The LSq estimation method was firstly used by Swain et al. [28] as a nonlinear method in estimation of the parameters of the Beta distribution. Especially, when the maximum likelihood estimators cannot be obtained in an explicit form, the LSq estimates are quite important with regard to provide an initial estimation for numerical methods which use in obtaining the maximum likelihood estimations.

The LSq estimations of the parameters α, β and λ, say $\hat{\alpha}_{LSq}$, $\hat{\beta}_{LSq}$ and $\hat{\lambda}_{LSq}$, respectively, are obtained by minimizing the equation

$$\sum_{i=1}^{n}\left(F_{APGR}(x_{(i)},\alpha,\beta,\lambda)-\hat{F}\left(x_{(i)}\right)\right)^2, \tag{37}$$

with respect to parameters α, λ and β, where $x_{(i)}$, $(i = 1, 2, \ldots, n)$ is the ith element of the ordered observations $x_1, x_2, \ldots, x_n$ and $\hat{F}(.)$ is indicate the observations' empirical cumulative distribution function (ecdf) calculated as

$$\hat{F}(.) = \frac{i}{n+1}. \tag{38}$$

By using the equations (4) and (38) in equation (37), we have

$$\sum_{i=1}^{n}\left(\frac{\alpha^{\left(1-e^{-\left(\lambda x_{(i)}\right)^2}\right)^{\beta}}-1}{\alpha-1}-\frac{i}{n+1}\right)^2. \tag{39}$$

Note that both LSq estimates and ML estimates of the unknown parameters can be obtained using the numerical methods.

5 Monte-Carlo Simulation Study

In this section, some simulation studies are presented in order to compare the estimation efficiencies of the ML and LSq estimators obtained in the previous section. In the simulation studies, two different cases $\alpha < 1$ and $\alpha > 1$ are considered.

In the first case, the parameter α is chosen as 0.25 and also the values of the parameters β, λ are set as $\beta = 0.5, 1, 2$ and $\lambda = 0.5, 1, 2$, respectively. The ML and LSq estimates of the parameters (Est.) are obtained with the simulations performed by 1000 replications for the different sample of sizes $n = 30, 50, 100$ and

Table 1: Parameter estimates, Bias and MSE values, when $\alpha = 0.25$

					α			β			λ		
α	β	λ	n	Method	Est.	Bias	MSE	Est.	Bias	MSE	Est	Bias	MSE
0.25	0.50	0.50	30	ML	0.3310	0.0810	0.0829	0.5339	0.0339	0.0128	0.5331	0.0331	0.0122
				LSq	0.4547	0.2047	0.1884	0.4843	-0.0157	0.0146	0.4960	-0.0040	0.0227
			50	ML	0.2985	0.0485	0.0511	0.5237	0.0237	0.0069	0.5234	0.0234	0.0071
				LSq	0.4221	0.1721	0.1561	0.4893	-0.0107	0.0098	0.4928	-0.0072	0.0143
			100	ML	0.2640	0.0140	0.0104	0.5094	0.0094	0.0031	0.5103	0.0103	0.0028
				LSq	0.4055	0.1555	0.1303	0.4845	-0.0155	0.0061	0.5023	0.0023	0.0104
			200	ML	0.2635	0.0135	0.0056	0.5038	0.0038	0.0012	0.5059	0.0059	0.0013
				LSq	0.3845	0.1345	0.0827	0.4827	-0.0173	0.0033	0.5115	0.0115	0.0057
		1.00	30	ML	0.3224	0.0724	0.0971	0.5268	0.0268	0.0118	1.0522	0.0522	0.0487
				LSq	0.4221	0.1721	0.1840	0.4879	-0.0121	0.0159	0.9644	-0.0356	0.0931
			50	ML	0.2905	0.0405	0.0494	0.5160	0.0160	0.0067	1.0317	0.0317	0.0264
				LSq	0.4396	0.1896	0.1731	0.4760	-0.0240	0.0085	0.9901	-0.0099	0.0762
			100	ML	0.2687	0.0187	0.0190	0.5093	0.0093	0.0037	1.0062	0.0062	0.0100
				LSq	0.4172	0.1672	0.1409	0.4834	-0.0166	0.0059	0.9966	-0.0034	0.0457
			200	ML	0.2560	0.0060	0.0054	0.5059	0.0059	0.0015	1.0102	0.0102	0.0040
				LSq	0.3654	0.1154	0.0842	0.4888	-0.0112	0.0033	1.0143	0.0143	0.0280
		2.00	30	ML	0.3508	0.1008	0.1126	0.5311	0.0311	0.0123	2.1191	0.1191	0.2319
				LSq	0.4029	0.1529	0.1659	0.4881	-0.0119	0.0133	1.8829	-0.1171	0.4014
			50	ML	0.3233	0.0733	0.0590	0.5125	0.0125	0.0064	2.0609	0.0609	0.1113
				LSq	0.4304	0.1804	0.1723	0.4836	-0.0164	0.0098	1.9363	-0.0637	0.2866
			100	ML	0.2771	0.0271	0.0219	0.5096	0.0096	0.0033	2.0237	0.0237	0.0397
				LSq	0.3729	0.1229	0.1164	0.4872	-0.0128	0.0054	1.9441	-0.0559	0.1902
			200	ML	0.2688	0.0188	0.0204	0.5019	0.0019	0.0016	2.0029	0.0029	0.0154
				LSq	0.3591	0.1091	0.0865	0.4881	-0.0119	0.0033	1.9935	-0.0065	0.1155
	1.00	0.50	30	ML	0.4193	0.1693	0.2044	1.0525	0.0525	0.0565	0.5234	0.0234	0.0108
				LSq	0.4877	0.2377	0.2023	0.9628	-0.0372	0.0845	0.5050	0.0050	0.0116
			50	ML	0.3653	0.1153	0.0830	1.0500	0.0500	0.0358	0.5268	0.0268	0.0064
				LSq	0.4403	0.1903	0.1651	0.9802	-0.0198	0.0439	0.5050	0.0050	0.0077
			100	ML	0.2990	0.0490	0.0339	1.0125	0.0125	0.0151	0.5074	0.0074	0.0032
				LSq	0.4238	0.1738	0.1423	0.9645	-0.0355	0.0226	0.5039	0.0039	0.0054
			200	ML	0.2732	0.0232	0.0093	1.0076	0.0076	0.0074	0.5056	0.0056	0.0013
				LSq	0.3890	0.1390	0.0929	0.9676	-0.0324	0.0127	0.5085	0.0085	0.0032
		1.00	30	ML	0.5775	0.3275	0.5573	1.0270	0.0270	0.0666	1.0490	0.0490	0.0547
				LSq	0.4907	0.2407	0.2161	0.9589	-0.0411	0.0817	0.9866	-0.0134	0.0540
			50	ML	0.4350	0.1850	0.2384	1.0246	0.0246	0.0356	1.0375	0.0375	0.0344
				LSq	0.4388	0.1888	0.1860	0.9794	-0.0206	0.0453	0.9958	-0.0042	0.0366
			100	ML	0.3179	0.0679	0.0481	1.0107	0.0107	0.0148	1.0133	0.0133	0.0168
				LSq	0.3941	0.1441	0.1358	0.9699	-0.0301	0.0218	0.9920	-0.0080	0.0250
			200	ML	0.2729	0.0229	0.0137	1.0123	0.0123	0.0077	1.0081	0.0081	0.0055
				LSq	0.3690	0.1190	0.1078	0.9780	-0.0220	0.0121	0.9955	-0.0045	0.0183
		2.00	30	ML	0.5230	0.2730	0.4866	1.0636	0.0636	0.0830	2.1203	0.1203	0.2022
				LSq	0.4464	0.1964	0.2029	0.9863	-0.0137	0.1071	1.9768	-0.0232	0.2354
			50	ML	0.4479	0.1979	0.2545	1.0163	0.0163	0.0358	2.0363	0.0363	0.1255
				LSq	0.4397	0.1897	0.1860	0.9671	-0.0329	0.0403	1.9618	-0.0382	0.1595
			100	ML	0.3651	0.1151	0.1078	1.0098	0.0098	0.0183	2.0244	0.0244	0.0798
				LSq	0.4319	0.1819	0.1677	0.9706	-0.0294	0.0226	1.9988	-0.0012	0.1115
			200	ML	0.2980	0.0480	0.0330	0.9982	-0.0018	0.0084	2.0104	0.0104	0.0308
				LSq	0.3954	0.1454	0.1261	0.9634	-0.0366	0.0138	1.9947	-0.0053	0.0728
	2.00	0.50	30	ML	0.6306	0.3806	1.1199	2.1752	0.1752	0.4756	0.5261	0.0261	0.0090
				LSq	0.4746	0.2246	0.1810	2.0750	0.0750	0.9362	0.5126	0.0126	0.0075
			50	ML	0.3896	0.1396	0.1791	2.0603	0.0603	0.1600	0.5053	0.0053	0.0055
				LSq	0.4653	0.2153	0.1629	1.9650	-0.0350	0.2443	0.5084	0.0084	0.0051
			100	ML	0.3223	0.0723	0.0492	2.0399	0.0399	0.0773	0.5042	0.0042	0.0030
				LSq	0.4474	0.1974	0.1374	1.9857	-0.0143	0.1249	0.5142	0.0142	0.0036
			200	ML	0.2839	0.0339	0.0170	2.0171	0.0171	0.0352	0.5033	0.0033	0.0011
				LSq	0.4171	0.1671	0.1012	1.9704	-0.0296	0.0536	0.5139	0.0139	0.0023
		1.00	30	ML	0.7320	0.4820	1.5364	2.1230	0.1230	0.3462	1.0528	0.0528	0.0346
				LSq	0.4745	0.2245	0.1941	1.9797	-0.0203	0.4356	1.0050	0.0050	0.0277
			50	ML	0.5323	0.2823	0.5206	2.0073	0.0073	0.1550	1.0196	0.0196	0.0258
				LSq	0.4783	0.2283	0.1855	1.9078	-0.0922	0.2113	1.0075	0.0075	0.0182
			100	ML	0.3521	0.1021	0.0891	2.0137	0.0137	0.0709	1.0044	0.0044	0.0154
				LSq	0.4389	0.1889	0.1481	1.9450	-0.0550	0.1009	1.0155	0.0155	0.0134
			200	ML	0.2776	0.0276	0.0239	2.0078	0.0078	0.0342	0.9966	-0.0034	0.0073
				LSq	0.4016	0.1516	0.1105	1.9548	-0.0452	0.0511	1.0155	0.0155	0.0105
		2.00	30	ML	0.7357	0.4857	1.4241	2.1062	0.1062	0.2895	2.1069	0.1069	0.1367
				LSq	0.4806	0.2306	0.2067	2.0005	0.0005	0.4792	2.0186	0.0186	0.1419
			50	ML	0.4790	0.2290	0.3525	2.0337	0.0337	0.1646	2.0350	0.0350	0.0762
				LSq	0.4346	0.1846	0.1688	1.9244	-0.0756	0.2082	1.9907	-0.0093	0.0667
			100	ML	0.3738	0.1238	0.1281	2.0218	0.0218	0.0851	2.0085	0.0085	0.0663
				LSq	0.4098	0.1598	0.1439	1.9477	-0.0523	0.1132	1.9959	-0.0041	0.0600
			200	ML	0.3109	0.0609	0.0568	2.0054	0.0054	0.0404	1.9894	-0.0106	0.0388
				LSq	0.4122	0.1622	0.1380	1.9563	-0.0437	0.0543	2.0094	0.0094	0.0498

200. In addition, through the simulation study, the bias (Bias) and mean-squared error (MSE) values of the ML and LSq estimators are obtained. The simulated results are given in Table 1.

For the second case of the simulation study, the α parameter is set as 4. Also, the values of the parameters β and λ are chosen $\beta = 0.5, 1, 2$ and $\lambda = 0.5, 1, 2$, respectively, as in the firs case. The simulated results are given by Table 2.

When the results given by Tables 1 and 2 are examined, it is seen that as the sample size n increases, both the estimations are close to actual values of the parameters and the ML and LSq estimators have smaller bias and MSE values for all cases. Furthermore, for both cases, it is concluded that the ML estimators outperform the LSq estimators with smaller MSE values according to the results given in Tables 1 and 2.

6 Application to Real Data

In this section, we present an analysis on a real-life data set called the coal mining disaster data set to illustrate the modeling behavior of the $APGR$ distribution in comparison with Rayleigh and generalized Rayleigh distributions. The data set includes 191 observation dealing with the intervals in days between successive coal mining disasters in Great Britain [29].

Firstly, we investigate the underlying distribution of the data set. We apply the Kolmogorov-Smirnov (KS) test statistic to check whether this data set follows the $APGR$ and most popular lifetime distributions such as Rayleigh, generalized Rayleigh, exponential, Gamma, Weibull, Log-Normal. The computed values of the KS statistic and corresponding p-values for each model are tabulated in Table 3.

By Table 3, we can say that the underlying distribution of the coal mining disaster data set is compatible with the $APGR$, Gamma, Weibull and Log-Normal distributions.

Now, we apply the $APGR$, Gamma, Weibull and Log-Normal distributions as a model to coal mining disaster data set and obtain the negative log-likelihood (Neg. Log-Lik) and Akaike information criterion (AIC) values for deciding the optimal distribution model to this data set. The ML and LSq estimations of the parameters with the obtained AIC and Neg. Log-Lik values are summarized in Table 4.

According to Table 4, it is concluded that the $APGR$ distribution gives the better fit to the dataset than the Weibull, Gamma and Log-Normal distributions since it has smaller AIC and Neg. Log-Lik values. The data fitting performance of the $APGR$ distribution can be clearly seen from Figure 2, which plots the ecdf and the cdf fitted by $APGR$ distribution. As can be seen from Figure 2, the fitted cdf strongly follows the empirical cdf of the observations and this is the desired case in real-life applications.

7 Conclusion

In this study, a new life-time distribution named the $APGR$ distribution is introduced. The pdf and cdf of the introduced distribution are derived using the APT method. The behavior of the pdf of $APGR$ distribution is displayed in Figure 1 for different values of the model parameters. The expressions for basic characteristics of the $APGR$ distribution such as hazard function, survival function, moments, characteristic function, skewness, kurtosis, order statistics, Shannon entropy, and stress-strength probability and Lorenz and Bonferroni curves are derived in the paper. Also, the estimators of the model parameters α, β and λ are obtained using two different methods the ML and LSq. The efficiencies of the ML and LSq estimators are also compared by comprehensive simulation studies on the different sample of sizes small, moderate and large. The simulation results show that the efficiencies of both estimators are quite satisfactory according to bias and MSE criteria for all sample sizes. Further, the ML and LSq estimators are asymptotically unbiased and consistent since, when the sample size increases, both bias and MSE values converge to zero.

The $APGR$ distribution presents better fit to the coal mining disaster data than Gamma, Weibull and Log-Normal distributions, with the smaller Neg. Log-Lik. and AIC values. Thus, we can say that the $APGR$ distribution provides the quite preferable modeling performance for life-time data and is a powerful alternative to the

Table 2: Parameter estimates, Bias and MSE values, when $\alpha = 4$

					α			β			λ		
α	β	λ	n	Method	Est.	Bias	MSE	Est.	Bias	MSE	Est	Bias	MSE
4.00	0.50	0.50	30	ML	4.0241	0.0241	0.2554	0.5273	0.0273	0.0131	0.5177	0.0177	0.0037
				LSq	4.0346	0.0346	0.2556	0.5320	0.0320	0.0187	0.5174	0.0174	0.0048
			50	ML	4.0299	0.0299	0.2391	0.5143	0.0143	0.0074	0.5063	0.0063	0.0020
				LSq	4.0325	0.0325	0.2399	0.5149	0.0149	0.0089	0.5068	0.0068	0.0025
			100	ML	4.0130	0.0130	0.0475	0.5047	0.0047	0.0021	0.5020	0.0020	0.0007
				LSq	4.0133	0.0133	0.0475	0.5045	0.0045	0.0024	0.5014	0.0014	0.0009
			200	ML	4.0009	0.0009	0.0075	0.5029	0.0029	0.0010	0.5034	0.0034	0.0004
				LSq	4.0012	0.0012	0.0075	0.5028	0.0028	0.0012	0.5032	0.0032	0.0005
		1.00	30	ML	4.0532	0.0532	0.4908	0.5340	0.0340	0.0151	1.0276	0.0276	0.0128
				LSq	4.0941	0.0941	0.4905	0.5595	0.0595	0.0304	1.0528	0.0528	0.0219
			50	ML	4.0322	0.0322	0.2618	0.5182	0.0182	0.0091	1.0166	0.0166	0.0062
				LSq	4.0597	0.0597	0.2865	0.5225	0.0225	0.0113	1.0234	0.0234	0.0092
			100	ML	4.0120	0.0120	0.0412	0.5065	0.0065	0.0026	1.0092	0.0092	0.0026
				LSq	4.0184	0.0184	0.0416	0.5089	0.0089	0.0041	1.0117	0.0117	0.0038
			200	ML	4.0071	0.0071	0.0219	0.5044	0.0044	0.0018	1.0041	0.0041	0.0014
				LSq	4.0083	0.0083	0.0219	0.5057	0.0057	0.0024	1.0030	0.0030	0.0021
		2.00	30	ML	4.0844	0.0844	0.4919	0.5378	0.0378	0.0172	2.0478	0.0478	0.0348
				LSq	4.1138	0.1138	0.4949	0.5550	0.0550	0.0272	2.0749	0.0749	0.0485
			50	ML	4.0352	0.0352	0.1014	0.5319	0.0319	0.0099	2.0316	0.0316	0.0215
				LSq	4.0524	0.0524	0.0986	0.5343	0.0343	0.0115	2.0414	0.0414	0.0281
			100	ML	4.0074	0.0074	0.0515	0.5117	0.0117	0.0038	2.0078	0.0078	0.0066
				LSq	4.0127	0.0127	0.0488	0.5132	0.0132	0.0044	2.0138	0.0138	0.0096
			200	ML	3.9978	-0.0022	0.0147	0.5061	0.0061	0.0014	2.0008	0.0008	0.0036
				LSq	3.9991	-0.0009	0.0148	0.5070	0.0070	0.0018	2.0014	0.0014	0.0049
	1.00	0.50	30	ML	4.0057	0.0057	0.0471	1.0075	0.0075	0.0254	0.5048	0.0048	0.0020
				LSq	4.0122	0.0122	0.0502	1.0173	0.0173	0.0379	0.5073	0.0073	0.0024
			50	ML	4.0038	0.0038	0.0579	1.0181	0.0181	0.0167	0.5042	0.0042	0.0011
				LSq	4.0060	0.0060	0.0585	1.0201	0.0201	0.0185	0.5052	0.0052	0.0012
			100	ML	3.9978	-0.0022	0.0080	1.0080	0.0080	0.0054	0.5013	0.0013	0.0005
				LSq	3.9983	-0.0017	0.0080	1.0083	0.0083	0.0057	0.5011	0.0011	0.0006
			200	ML	4.0006	0.0006	0.0039	1.0016	0.0016	0.0022	0.5007	0.0007	0.0003
				LSq	4.0007	0.0007	0.0039	1.0018	0.0018	0.0023	0.5007	0.0007	0.0003
		1.00	30	ML	3.9814	-0.0186	0.0329	1.0426	0.0426	0.0724	1.0036	0.0036	0.0069
				LSq	4.0030	0.0030	0.0849	1.0489	0.0489	0.1037	1.0116	0.0116	0.0088
			50	ML	4.0021	0.0021	0.0313	1.0150	0.0150	0.0242	1.0089	0.0089	0.0042
				LSq	4.0105	0.0105	0.0325	1.0157	0.0157	0.0294	1.0074	0.0074	0.0054
			100	ML	3.9991	-0.0009	0.0323	1.0040	0.0040	0.0117	1.0027	0.0027	0.0020
				LSq	4.0001	0.0001	0.0323	1.0044	0.0044	0.0124	1.0020	0.0020	0.0024
			200	ML	3.9991	-0.0009	0.0036	1.0068	0.0068	0.0042	1.0014	0.0014	0.0011
				LSq	3.9998	-0.0002	0.0037	1.0061	0.0061	0.0045	1.0012	0.0012	0.0012
		2.00	30	ML	3.9995	-0.0005	0.0347	1.0524	0.0524	0.0644	2.0090	0.0090	0.0177
				LSq	4.0183	0.0183	0.0349	1.0597	0.0597	0.0691	2.0163	0.0163	0.0255
			50	ML	3.9938	-0.0062	0.0120	1.0406	0.0406	0.0519	2.0138	0.0138	0.0124
				LSq	4.0082	0.0082	0.0144	1.0562	0.0562	0.0569	2.0202	0.0202	0.0212
			100	ML	4.0014	0.0014	0.0039	1.0130	0.0130	0.0172	2.0056	0.0056	0.0045
				LSq	4.0049	0.0049	0.0047	1.0161	0.0161	0.0196	2.0091	0.0091	0.0071
			200	ML	3.9990	-0.0010	0.0020	1.0144	0.0144	0.0076	2.0044	0.0044	0.0020
				LSq	4.0001	0.0001	0.0020	1.0155	0.0155	0.0088	2.0046	0.0046	0.0024
	2.00	0.50	30	ML	4.0007	0.0007	0.0692	2.0598	0.0598	0.1088	0.5010	0.0010	0.0011
				LSq	4.0007	0.0007	0.0692	2.0598	0.0598	0.1089	0.5012	0.0012	0.0011
			50	ML	4.0149	0.0149	0.0601	2.0287	0.0287	0.0718	0.5033	0.0033	0.0007
				LSq	4.0151	0.0151	0.0602	2.0293	0.0293	0.0719	0.5032	0.0032	0.0007
			100	ML	4.0008	0.0008	0.0145	2.0182	0.0182	0.0212	0.5028	0.0028	0.0004
				LSq	4.0008	0.0008	0.0145	2.0182	0.0182	0.0212	0.5028	0.0028	0.0004
			200	ML	4.0008	0.0008	0.0179	2.0033	0.0033	0.0086	0.5003	0.0003	0.0002
				LSq	4.0008	0.0008	0.0179	2.0033	0.0033	0.0086	0.5003	0.0003	0.0002
		1.00	30	ML	4.0163	0.0163	0.2120	2.1333	0.1333	0.3086	1.0133	0.0133	0.0056
				LSq	4.0181	0.0181	0.2113	2.1332	0.1332	0.3090	1.0149	0.0149	0.0062
			50	ML	4.0004	0.0004	0.0454	2.0808	0.0808	0.1425	1.0084	0.0084	0.0025
				LSq	4.0012	0.0012	0.0448	2.0807	0.0807	0.1424	1.0095	0.0095	0.0027
			100	ML	3.9903	-0.0097	0.0398	2.0413	0.0413	0.0636	1.0036	0.0036	0.0014
				LSq	3.9902	-0.0098	0.0398	2.0404	0.0404	0.0637	1.0043	0.0043	0.0014
			200	ML	3.9978	-0.0022	0.0125	2.0100	0.0100	0.0137	1.0017	0.0017	0.0006
				LSq	3.9979	-0.0021	0.0125	2.0103	0.0103	0.0137	1.0014	0.0014	0.0006
		2.00	30	ML	4.0296	0.0296	0.1536	2.1908	0.1908	0.6677	2.0249	0.0249	0.0206
				LSq	4.0440	0.0440	0.1859	2.1871	0.1871	0.6694	2.0276	0.0276	0.0219
			50	ML	3.9930	-0.0070	0.0438	2.0927	0.0927	0.2209	2.0126	0.0126	0.0095
				LSq	3.9937	-0.0063	0.0436	2.0908	0.0908	0.2225	2.0138	0.0138	0.0099
			100	ML	3.9876	-0.0124	0.0250	2.0371	0.0371	0.0760	2.0054	0.0054	0.0046
				LSq	3.9892	-0.0108	0.0221	2.0368	0.0368	0.0760	2.0061	0.0061	0.0046
			200	ML	3.9912	-0.0088	0.0140	2.0192	0.0192	0.0337	2.0013	0.0013	0.0022
				LSq	3.9915	-0.0085	0.0138	2.0194	0.0194	0.0337	2.0013	0.0013	0.0022

Table 3: KS test results of the possible models for the coal mining disaster data set.

	Model						
	APGR	Rayleigh	Generalized Rayleigh	Exponential	Weibull	Gamma	Log-Normal
KS	0.0370	0.4530	0.1531	0.1040	0.0464	0.0558	0.0742
p-value	0.9483	7.85E-35	2.36E-04	0.0340	0.7907	0.5753	0.2352

Table 4: Model comparison and parameter estimates for the coal mining disaster data set.

	Model							
	APGR		Weibull		Gamma		Log-Normal	
Neg. Log-Lik.	1197.6		1198.6		1201.4		1204.2	
AIC	2401.1		2401.2		2406.8		2412.3	
ML Estimations	α	0.0045	α_W	184.8301	α_G	0.7211	μ_{LN}	4.5286
	β	0.4134	θ_W	0.7928	β_G	295.9621	σ_{LN}	1.4772
	λ	0.0007						

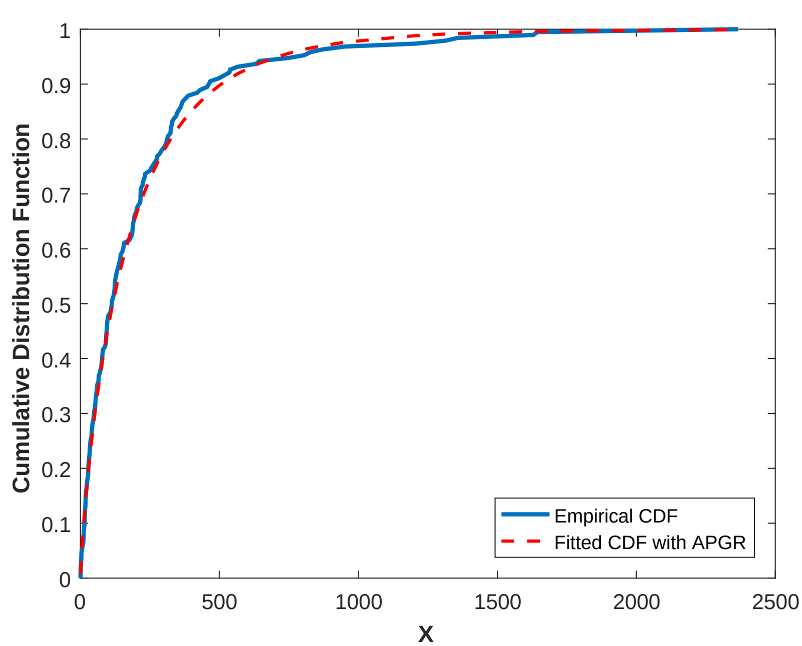

Figure 2: For the coal mining disaster data set, empirical and the fitted cdf with $APGR$ distribution.

famous life-time distributions such as Gamma, Weibull and Log-Normal. Further, by information from real data application carried out using the coal mining disaster data set, it can be said that the $APGR$ distribution has displayed more flexible data modeling performance than the baseline distributions Generalized Rayleigh and Rayleigh. Because while the $APGR$ distribution is a suitable model for the coal mining disaster data set according to the obtained results of the KS test statistic given in Table 3, the Generalized Rayleigh and Rayleigh distributions aren't appropriate models. Therefore, it can be said that the $APGR$ distribution has capable of modeling more data types than the baseline distributions generalized Rayleigh and Rayleigh.

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A Appendix: Proof of Lemma 1

By using the power expansion formula, equation (11) can be written as

$$\begin{aligned}
\xi(a,b,r,L,\delta) &= \int_0^\infty x^{r+1} e^{-(Lx)^2}\left(1-e^{-(Lx)^2}\right)^{b-1} a^{\left(1-e^{-(Lx)^2}\right)^b} e^{\delta x}\,dx \\
&= \sum_{i=0}^\infty \frac{(\log a)^i}{i!}\int_0^\infty x^{r+1} e^{-(Lx)^2}\left(1-e^{-(Lx)^2}\right)^{b-1}\left[\left(1-e^{-(Lx)^2}\right)^b\right]^i e^{\delta x}\,dx \\
&= \sum_{i=0}^\infty\sum_{j=0}^{ib}\binom{ib}{j}(-1)^j\frac{(\log a)^i}{i!}\int_0^\infty x^{r+1} e^{-(Lx)^2}\left(1-e^{-(Lx)^2}\right)^{b-1}\left[e^{-(Lx)^2}\right]^j e^{\delta x}\,dx \\
&= \sum_{i=0}^\infty\sum_{j=0}^{ib}\sum_{k=0}^{b-1}\binom{ib}{j}\binom{b-1}{k}(-1)^j(-1)^k\frac{(\log a)^i}{i!}\int_0^\infty x^{r+1} e^{-(Lx)^2}\left[e^{-(Lx)^2}\right]^k\left[e^{-(Lx)^2}\right]^j e^{\delta x}\,dx \\
&= \sum_{i=0}^\infty\sum_{j=0}^{ib}\sum_{k=0}^{b-1}\binom{ib}{j}\binom{b-1}{k}(-1)^j(-1)^k\frac{(\log a)^i}{i!}\int_0^\infty x^{r+1} e^{-(Lx)^2} e^{-k(Lx)^2} e^{-j(Lx)^2} e^{\delta x}\,dx \\
&= \sum_{i=0}^\infty\sum_{j=0}^{ib}\sum_{k=0}^{b-1}\binom{ib}{j}\binom{b-1}{k}(-1)^{k+j}\frac{(\log a)^i}{i!}\int_0^\infty x^{r+1} e^{-(Lx)^2(k+j+1)} e^{\delta x}\,dx
\end{aligned} \tag{40}$$

and by applying the gamma function in the last equation, we have

$$\begin{aligned}
\xi(a,b,L,r,\delta) = \sum_{i=0}^\infty\sum_{j=0}^{ib}\sum_{k=0}^{b-1}&\left\{\frac{(\log a)^i}{i!}(-1)^{j+k}\binom{ib}{j}\binom{b-1}{k}\frac{1}{2}\left(L^2(j+k+1)\right)^{\frac{1}{2}(-r-3)}\times\right.\\
&\left(\delta\Gamma\left(\frac{r+3}{2}\right)\,{}_1F_1\left(\frac{r+3}{2};\frac{3}{2};\frac{\delta^2}{4(j+k+1)L^2}\right)+\right.\\
&\left.\left.\Gamma\left(\frac{r}{2}+1\right)\sqrt{L^2(j+k+1)}\,{}_1F_1\left(\frac{r+2}{2};\frac{1}{2};\frac{\delta^2}{4(j+k+1)L^2}\right)\right)\right\}
\end{aligned} \tag{41}$$

B Appendix: Calculation of the Rényi entropy of the $APGR$ distribution.

The Rényi entropy of the $APGR$ distribution is

$$RE_X(\xi) = \frac{1}{1-\xi}\ln\int_0^\infty (f(x))^\xi\,dx$$

$$= \frac{1}{1-\xi} \ln \int_0^\infty \left(\frac{\ln \alpha}{\alpha - 1} 2\beta\lambda^2 x e^{-(\lambda x)^2} \left(1 - e^{-(\lambda x)^2}\right)^{\beta-1} \alpha^{\left(1-e^{-(\lambda x)^2}\right)^\beta} \right)^\xi dx$$

$$= \frac{1}{1-\xi} \ln \left(\left(\frac{2\beta\lambda^2 \ln \alpha}{\alpha - 1} \right)^\xi \int_0^\infty \left(x e^{-(\lambda x)^2} \left(1 - e^{-(\lambda x)^2}\right)^{\beta-1} \alpha^{\left(1-e^{-(\lambda x)^2}\right)^\beta} \right)^\xi \right) dx. \tag{42}$$

Applying the power expansion formula, the equation (42) is written as

$$RE_X(\xi) = \frac{1}{1-\xi} \ln \left(\left(\frac{2\beta\lambda^2 \ln \alpha}{\alpha - 1} \right)^\xi \int_0^\infty \left(x e^{-(\lambda x)^2} \left(1 - e^{-(\lambda x)^2}\right)^{\beta-1} \alpha^{\left(1-e^{-(\lambda x)^2}\right)^\beta} \right)^\xi \right) dx$$

$$= \frac{1}{1-\xi} \ln \left(\left(\frac{2\beta\lambda^2 \ln \alpha}{\alpha - 1} \right)^\xi \int_0^\infty \left(x e^{-(\lambda x)^2} \left(1 - e^{-(\lambda x)^2}\right)^{\beta-1} \sum_{i=0}^\infty \frac{(\ln \alpha)^i}{i!} \left(1 - e^{-(\lambda x)^2}\right)^{i\beta} \right)^\xi \right) dx$$

$$= \frac{1}{1-\xi} \ln \left(\left(\frac{2\beta\lambda^2 \ln \alpha}{\alpha - 1} \right)^\xi \sum_{i=0}^\infty \frac{(\ln \alpha)^i}{i!} \int_0^\infty \left(x e^{-(\lambda x)^2} \left(1 - e^{-(\lambda x)^2}\right)^{\beta-1} \left(1 - e^{-(\lambda x)^2}\right)^{i\beta} \right)^\xi \right) dx$$

$$= \frac{1}{1-\xi} \ln \left(\left(\frac{2\beta\lambda^2 \ln \alpha}{\alpha - 1} \right)^\xi \sum_{i=0}^\infty \frac{(\ln \alpha)^i}{i!} \int_0^\infty \left(x e^{-(\lambda x)^2} \left(1 - e^{-(\lambda x)^2}\right)^{\beta-1+i\beta} \right)^\xi \right) dx$$

$$= \frac{1}{1-\xi} \ln \left(\left(\frac{2\beta\lambda^2 \ln \alpha}{\alpha - 1} \right)^\xi \sum_{i=0}^\infty \frac{(\ln \alpha)^i}{i!} \int_0^\infty \left(x e^{-(\lambda x)^2} \sum_{j=0}^{\beta-1+i\beta} \binom{\beta - 1 + i\beta}{j} (-1)^j \left(e^{-(\lambda x)^2}\right)^j \right)^\xi \right) dx$$

$$= \frac{1}{1-\xi} \ln \left(\left(\frac{2\beta\lambda^2 \ln \alpha}{\alpha - 1} \right)^\xi \sum_{i=0}^\infty \frac{(\ln \alpha)^i}{i!} \sum_{j=0}^{\beta-1+i\beta} \binom{\beta - 1 + i\beta}{j} (-1)^j \int_0^\infty \left(x e^{-(\lambda x)^2} \left(e^{-(\lambda x)^2}\right)^j \right)^\xi \right) dx$$

$$= \frac{1}{1-\xi} \ln \left(\left(\frac{2\beta\lambda^2 \ln \alpha}{\alpha - 1} \right)^\xi \sum_{i=0}^\infty \frac{(\ln \alpha)^i}{i!} \sum_{j=0}^{\beta-1+i\beta} \binom{\beta - 1 + i\beta}{j} (-1)^j \int_0^\infty x^\xi e^{-\xi(j+1)(\lambda x)^2} \right) dx \tag{43}$$

By applying the gamma function to equation (43), we have

$$RE_X(\xi) = \frac{1}{1-\xi} \ln \left(\left(\frac{2\beta\lambda^2 \ln \alpha}{\alpha - 1} \right)^\xi \sum_{i=0}^\infty \frac{(\ln \alpha)^i}{i!} \sum_{j=0}^{i\beta+\beta-1} \binom{i\beta + \beta - 1}{j} (-1)^j \frac{1}{2} \Gamma\left(\frac{\xi + 1}{2}\right) \left((j+1)\lambda^2 \xi\right)^{\frac{1}{2}(-\xi-1)} \right) \tag{44}$$